Improved rate for a multi-server coded caching

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Abstract

Multi-server coded caching, which can further reduce the amount of transmission by means of the collaboration among these servers in the wireless network during the peak traffic times, can be seen everywhere in our life. This amount is called rate. The three servers setting (two data servers and one parity check server)is widely used in practice, e.g. redundant array of independent disks-4. Luo et al. in 2016 proposed the first coded caching scheme for this setting. In this paper, we obtain a smaller rate by modifying Luo's scheme for some parameters. In addition, our method can be more generalized to further reduce the rate.

I. Introduction

Predominantly driven by video content demand, there is a dramatic increase in wireless traffic now. The high temporal variability of network traffic results in communication systems that are congested during peak-traffic times and underutilized during off-peak times. Caching is natural strategy to cope with this high temporal variability by shifting some transmissions from peak to off-peak times with the help of cache memories at the network edge.

Maddah-Ali and Niesen in [6] proved that coded caching does not only shift some transmissions from peak to off-peak times, but also further reduces the amount of transmission during the peak traffic times by exploiting caches to create multicast opportunities. The first caching scenario is focused in [6]: a single server containing N files with the same length connects to K users over a shared link and each user has a cache memory of size M files. During the off-peak traffic times the server places some contents to each user's cache. In this phase the server does not known what each user will require next. During the peak traffic times, each user requires a file from server randomly. Then according to each user's cache, the server sends a coded signal (XOR of some required packets) to the users such that various user demands are satisfied. The first determined coded caching scheme, which is called MN scheme in this paper, was proposed in [6]. It is worth mentioning that the broadcasted amount of MN scheme for the worst request, where all the requirements are different from each other, is at most four times larger than the lower bound when $K \leq N$ [2]. We denote such amount by $R_{MN}(K, \frac{M}{N})$. So MN scheme has been extensively employed in practical scenarios, such as device to device networks [3], hierarchical networks [4], security [8], multi-servers setting [5], [7], [9] and so on. There are also many results following MN scheme in [2], [10], [11], [14]–[17] etc.

The coded caching used in muti-server setting can be seen everywhere. We focus on the setting in [5] which is also widely used (e.g. redundant array of independent disks-4) in our life. In this setting there are three servers, i.e., two data servers A, B storing $N/2$ disjoint files respectively and a parity server P storing the bitwise XOR of the information in A and B. The servers connect to users and operate on independent errorfree channels. This implies that these servers can transmit messages simultaneously and without interference to the same or different users. According to user's requirements, each server combines multiple segments from its own files into a single message, and broadcasts it such that each user can be satisfied by means of its cache and the received signal messages from servers.

In practice servers are aware of the content cached by each user and of the content stored in other servers. So even though any two files sorted on different servers can not be combined into a single message, the servers can still coordinate the messages of these two files. Denote the maximum amount broadcasted among the three servers by R files for the worst request. Clearly it is meaningful to make R as small as possible. Luo et al., in [5] constructed the first determined coded caching scheme by means of MN scheme and the results on saturating matching in bipartite graph. Specifically they first considered the symmetric request, i.e., both data servers receive the same number of requests, and showed that in their scheme the rate $R = \frac{1}{2}R_{MN}(K, \frac{M}{N})$

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if $K\frac{M}{N}$ is even, otherwise $R=(\frac{1}{2}+\frac{1}{6}\Delta(\frac{M}{N}))R_{MN}(K,\frac{M}{N})$ where the upper bound of $\Delta(\frac{M}{N})$ is $\frac{1}{3}$. Then a scheme and the related rate for the other requests can be obtained directly by means of several classes of schemes in symmetric requests.

In this paper, by modifying the schemes in [5] we derive a new rate $R=(\frac{1}{2}+\frac{1}{6}\Delta'(\frac{M}{N}))R_{MN}(K,\frac{M}{N})$ when $K\frac{M}{N}$ is odd. Here $\Delta'(\frac{M}{N})$ is obviously smaller than $\Delta(\frac{M}{N})$ in most cases. In particular K is large, $\frac{\Delta'(M/N)}{\Delta(M/N)}\approx 0$ if $\frac{M}{N}$ nears $1/2$, and $\frac{\Delta'(M/N)}{\Delta(M/N)}\approx\frac{1}{9}$ if $\frac{M}{N}$ nears $\frac{1}{3}$ or $\frac{2}{3}$. The rest of this paper is organized as follows. Section II briefly reviews MN scheme, the scheme proposed in [5] and the related concepts. In Section III, an improved scheme and its performance analysis are proposed. Conclusion is drawn in Section IV.

II. Preliminaries

A. MN scheme and bipartite graph

In a (K,M,N) caching system, denote N files by W_1, W_2, …, W_N. In the single server setting, when $t=K\frac{M}{N}$ is an integer, an MN scheme can be realized as follows [6].

- During the off-peak traffic times, each file is divided into $F=\binom{K}{t}$ equal packets, and is denoted by $W_i=\{W_{i,\mathcal{T}}\mid\mathcal{T}\subseteq[1,K],|\mathcal{T}|=t\}$. Each use k caches the following packets.
$$\mathcal{Z}_k=\{W_{i,\mathcal{T}}\mid k\in\mathcal{T},i=1,2,\ldots,N\}$$
- During the peak traffic times, each user requires a file randomly. Denote the user request by $\mathbf{d}=(d_1,d_2,\ldots,d_K)$. Then for each subset $\mathcal{S}$ with cardinality $t+1$ of $[1,K]$, the server broadcasts the following coded signal.
$$\bigoplus_{k\in\mathcal{S}}W_{d_k,\mathcal{S}\setminus\{k\}} \qquad (1)$$
Clearly the server broadcasts $\binom{K}{t+1}$ times. So the amount of transmission by server is
$$R_{MN}(K,\tfrac{M}{N})=\binom{K}{t+1}\Big/\binom{K}{t}=\frac{K-t}{t+1}.$$

The following concepts related on graph and matching are useful in this paper. A graph is denoted by $\mathbf{G}=(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges. A subset of edges $\mathcal{M}\subseteq\mathcal{E}$ is a matching if no two edges have a common vertex. A bipartite graph, denoted by $\mathbf{G}=(\mathcal{X},\mathcal{Y};\mathcal{E})$, is a graph whose vertices are divided into two disjoint parts $\mathcal{X}$ and $\mathcal{Y}$ such that every edge in $\mathcal{E}$ connects a vertex in $\mathcal{X}$ to one in $\mathcal{Y}$. For a set $X\subseteq\mathcal{X}$, let $N_{\mathbf{G}}(X)$ denote the set of all vertices in $\mathcal{Y}$ adjacent to some vertex of X. The degree of a vertex is the number of vertices adjacent to it. If every vertex of $\mathcal{X}$ has the same degree, we also call such a degree the degree of $\mathcal{X}$ and denote $d(\mathcal{X})$.

Theorem 1: (Hall's Marriage Theorem, [1]) Given a bipartite graph $\mathbf{G}=(\mathcal{X},\mathcal{Y};\mathcal{E})$, there exists a matching with $|\mathcal{X}|$ edges, i.e., a saturating matching, if and only if $|X|\leq|N_{\mathbf{G}}(X)|$ holds for any subset $X\subseteq\mathcal{X}$.

Corollary 1: ([1]) Given a bipartite graph $\mathbf{G}=(\mathcal{X},\mathcal{Y};\mathcal{E})$, assume that $d(\mathcal{X})=m$ and $d(\mathcal{Y})=n$. If $m\leq n$, then there is a saturating matching.

B. The scheme in [5]

Now let us consider the three servers setting. In the following we denote the files in server A and B by $\{A_1,\ldots,A_{N/2}\}$ and $\{B_1,\ldots,B_{N/2}\}$ respectively. So the files in parity server P are $\{A_1\oplus B_1,\ldots,A_{N/2}\oplus B_{N/2}\}$. That is Table I.

TABLE I
Files stored in each server

Server A	Server B	Server P
A_1	B_1	$A_1\oplus B_1$
A_2	B_2	$A_2\oplus B_2$
$\vdots$	$\vdots$	$\vdots$
$A_{N/2}$	$B_{N/2}$	$A_{N/2}\oplus B_{N/2}$

And the following notations are useful.

- Denote the set of users requesting files from server A by $\mathcal{K}_A$, and the set of users requesting files from server B by $\mathcal{K}_B$. Clearly the set of all the users is $\mathcal{K}=\mathcal{K}_A\bigcup\mathcal{K}_B$. Let $K_A=|\mathcal{K}_A|$ and $K_B=|\mathcal{K}_B|$.
- Assume that the kth user of $\mathcal{K}_A$ requests the d_k-th file in server A, and the kth user of $\mathcal{K}_B$ requests the d_k-th file in server B.

Luo et al., in [5] used the same caching strategy as MN scheme during the off-peak traffic times and modified the coded signals in (1) during the peak traffic times as follows. Given a subset $\mathcal{S}_1$ of size $t+1$, it can be divided into three parts, say $\mathcal{Q}_A$, $\mathcal{Q}_B$ and $\mathcal{Q}_A^{'}$ where $\mathcal{Q}_A,\mathcal{Q}_A^{'}\subseteq\mathcal{K}_A$ and $\mathcal{Q}_B\subseteq\mathcal{K}_B$. If there exists a subset $\mathcal{S}_2$ of size $t+1$ which can be divided into $\mathcal{Q}_A$, $\mathcal{Q}_B$ and $\mathcal{Q}_B^{'}$ where $\mathcal{Q}_B^{'}\subseteq\mathcal{K}_B$, then the pair $(\mathcal{S}_1,\mathcal{S}_2)$ is called an effective pair. When servers A, B and P broadcast the following messages respectively

$$\begin{aligned}
m^A_{\mathcal{S}_1} &= \left(\bigoplus_{k\in\mathcal{Q}_A} A_{d_k,\mathcal{S}_1\setminus\{k\}}\right)\bigoplus\left(\bigoplus_{k\in\mathcal{Q}_B} A_{d_k,\mathcal{S}_1\setminus\{k\}}\right)\bigoplus\left(\bigoplus_{k\in\mathcal{Q}'_A} A_{d_k,\mathcal{S}_1\setminus\{k\}}\right)\\
m^B_{\mathcal{S}_2} &= \left(\bigoplus_{k\in\mathcal{Q}_A} B_{d_k,\mathcal{S}_2\setminus\{k\}}\right)\bigoplus\left(\bigoplus_{k\in\mathcal{Q}_B} B_{d_k,\mathcal{S}_2\setminus\{k\}}\right)\bigoplus\left(\bigoplus_{k\in\mathcal{Q}'_B} B_{d_k,\mathcal{S}_2\setminus\{k\}}\right)\\
m^P_{\mathcal{S}_1\bigcap\mathcal{S}_2} &= \left[\bigoplus_{k\in\mathcal{Q}_B}\left(A_{d_k,\mathcal{S}_1\setminus\{k\}}\oplus B_{d_k,\mathcal{S}_1\setminus\{k\}}\right)\right]\bigoplus\left[\bigoplus_{k\in\mathcal{Q}_A}\left(B_{d_k,\mathcal{S}_2\setminus\{k\}}\oplus A_{d_k,\mathcal{S}_2\setminus\{k\}}\right)\right]\\
&= \left[\left(\bigoplus_{k\in\mathcal{Q}_B} A_{d_k,\mathcal{S}_1\setminus\{k\}}\right)\bigoplus\left(\bigoplus_{k\in\mathcal{Q}_B} B_{d_k,\mathcal{S}_1\setminus\{k\}}\right)\right]\bigoplus\left[\left(\bigoplus_{k\in\mathcal{Q}_A} B_{d_k,\mathcal{S}_2\setminus\{k\}}\right)\bigoplus\left(\bigoplus_{k\in\mathcal{Q}_A} A_{d_k,\mathcal{S}_2\setminus\{k\}}\right)\right]
\end{aligned} \tag{2}$$

Then each user in $\mathcal{S}_1$ and $\mathcal{S}_2$ can obtain the requested packets from $m^A_{\mathcal{S}_1}$, $m^B_{\mathcal{S}_2}$ and $m^P_{\mathcal{S}_1\bigcap\mathcal{S}_2}$. So if the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ form an effective pair, then the messages indexed by $\mathcal{S}_1$ and $\mathcal{S}_2$ in (1) can be replaced by three messages in (2). This implies that two messages, which should be transmitted, can be completed as each server just transmits a single message. Clearly we prefer to make as many effective pairs as possible. In the case of symmetric request, i.e., $K_A=K_B$, Luo et al., obtained the following results.

Theorem 2: ([5]) Based on MN scheme, when $K_A=K_B$ and $K=K_A+K_B$, the rate of the server system in Table I is

$$R_T(K,\tfrac{M}{N})=\begin{cases}\frac{1}{2}R_{MN}(K,\frac{M}{N}) & \text{if } K\frac{M}{N} \text{ is even}\\ (\frac{1}{2}+\frac{1}{6}\Delta(\frac{M}{N}))R_{MN}(K,\frac{M}{N}) & \text{if } K\frac{M}{N} \text{ is odd}\end{cases} \tag{3}$$

where $\Delta(\frac{M}{N})$, which is bounded by $\frac{1}{3}$, represents the ratio of unpaired messages.

In the following we will focus on the symmetric request. For the sake of introduction in this paper, we will use the following rules.

Remark 1: Each vertex of a graph is always represented by a subset $\mathcal{S}\subseteq\mathcal{K}$ with size $t+1$. And for any bipartite graph $\mathbf{G}=(\mathcal{X},\mathcal{Y};\mathcal{E})$ where a vertex $\mathcal{S}\in\mathcal{X}$ is adjacent to $\mathcal{S}'\in\mathcal{Y}$ if and only if they can form an effective pair.

C. Research motivation

The brief proof of Theorem 2 is very useful. Here we take the case that t is odd as an example. For each $w=0,1,\ldots,$ $t+1$, define

$$\mathcal{V}_w=\{\mathcal{S}\subseteq\mathcal{K}\ |\ |\mathcal{S}|=t+1,|\mathcal{S}\bigcap\mathcal{K}_A|=w\} \tag{4}$$

Clearly $|\mathcal{V}_w|=|\mathcal{V}_{t+1-w}|=\binom{K_A}{w}\binom{K_B}{t+1-w}=\binom{K_A}{t+1-w}\binom{K_B}{w}$ since $K_A=K_B$. By the fact $\binom{K}{t+1}=\sum_{w=0}^{t+1}\binom{K_A}{w}\binom{K_B}{t+1-w}$, Luo et. al. [5] constructed several classes of bipartite graphes satisfying Corollary 1 in the following way. For each $w\in[1,\frac{t-1}{2})$, they defined a bipartite graph $\mathbf{G}_w=(\mathcal{V}_w,\mathcal{V}_{t+1-w};\mathcal{E}_w)$ by (4) and showed that these bipartite graphes satisfy Corollary 1. When $K_A<t+1$, $\mathcal{S}\bigcap\mathcal{K}_A\neq\emptyset$ always holds for each subset $\mathcal{S}\subseteq\mathcal{K}$ with cardinality $t+1$. So they did not need to consider

the case $w=0$. When $K_A \geq t+1$ and $w=0$, assume that server A (and B) broadcast the messages $m^A_{\mathcal{S}}$ (and $m^B_{\mathcal{S}}$), $\mathcal{S}\subseteq\mathcal{K}_A$ (and $\mathcal{S}\subseteq\mathcal{K}_B$) independently. For the sets $\mathcal{V}_{\frac{t-1}{2}}$, $\mathcal{V}_{\frac{t+1}{2}}$, and $\mathcal{V}_{\frac{t+3}{2}}$, let

$$\mathcal{X}=\mathcal{V}_{\frac{t-1}{2}}\bigcup\mathcal{V}_{\frac{t+3}{2}}, \qquad \mathcal{Y}=\mathcal{V}_{\frac{t+1}{2}}. \tag{5}$$

They defined a bipartite graph $\mathbf{G}=(\mathcal{X},\mathcal{Y};\mathcal{E})$ and showed that there is a saturating matching. So the number of unpaired messages is

$$n=\left|\binom{K_A}{(t+1)/2}\binom{K_B}{(t+1)/2}-\binom{K_A}{(t-1)/2}\binom{K_B}{(t+3)/2}-\binom{K_B}{(t-1)/2}\binom{K_A}{(t+3)/2}\right| \tag{6}$$

and the ratio of unpaired messages is

$$\Delta(\frac{t}{K})=\frac{n}{\binom{K}{t+1}}. \tag{7}$$

Since each unpaired message can be transmitted by any two servers, each server could transmit $\frac{2}{3}n$ unpaired messages. So the rate is

$$R_T(K,\tfrac{M}{N})=\frac{[(1-\Delta(\frac{t}{K}))+\frac{2}{3}\Delta(\frac{t}{K})]\binom{K}{t+1}}{\binom{K}{t}}=\left(\frac{1}{2}+\frac{1}{6}\Delta(\frac{t}{K})\right)R_{MN}(K,\tfrac{M}{N}).$$

This is the result in Theorem 2. From the above discussions, it is sufficient to study the number of unpaired messages when we want to reduce rate $R_T(K,\frac{M}{N})$.

III. Improved scheme

In this section, we focus on the case of symmetric request for the case that t is odd. Clearly an intuitive approach to reduce the ratio of unpaired messages is finding the maximal matching of graph $\mathbf{G}=(\mathcal{V}_{\frac{t-1}{2}}\bigcup\mathcal{V}_{\frac{t+1}{2}}\bigcup\mathcal{V}_{\frac{t+3}{2}},\mathcal{E})$. It is well known that this maximal problem is an NP-hard and its complexity is very high since there are $\binom{K/2}{(t+1)/2}\binom{K/2}{(t+1)/2}+\binom{K/2}{(t-1)/2}\binom{K/2}{(t+3)/2}+\binom{K/2}{(t+3)/2}\binom{K/2}{(t-1)/2}$ vertices. We will propose a local maximal matching method to reduce the complexity.

Denote $\mathcal{K}_A=\{a_1,\,a_2,\,\ldots,\,a_{K_A}\}$ and $\mathcal{K}_B=\{b_1,\,b_2,\,\ldots,\,b_{K_B}\}$. We divide sets $\mathcal{V}_{\frac{t-1}{2}}$, $\mathcal{V}_{\frac{t+1}{2}}$ and $\mathcal{V}_{\frac{t+3}{2}}$ into four subsets respectively in the following way:

$$\begin{aligned}\mathcal{V}_{w;a_1,b_1}&=\{\mathcal{S}\in\mathcal{V}_w\mid a_1\in\mathcal{S},b_1\in\mathcal{S}\}, & \mathcal{V}_{w;a_1,\overline{b}_1}&=\{\mathcal{S}\in\mathcal{V}_w\mid a_1\in\mathcal{S},b_1\notin\mathcal{S}\},\\ \mathcal{V}_{w;\overline{a}_1,b_1}&=\{\mathcal{S}\in\mathcal{V}_w\mid a_1\notin\mathcal{S},b_1\in\mathcal{S}\}, & \mathcal{V}_{w;\overline{a}_1,\overline{b}_1}&=\{\mathcal{S}\in\mathcal{V}_w\mid a_1\notin\mathcal{S},b_1\notin\mathcal{S}\},\end{aligned} \tag{8}$$

where $w=\frac{t-1}{2}$, $\frac{t+1}{2}$, $\frac{t+3}{2}$. It is easy to check that

$$\begin{aligned}|\mathcal{V}_{w;a_1,b_1}|&=\binom{K/2-1}{w-1}\binom{K/2-1}{t-w}, & |\mathcal{V}_{w;a_1,\overline{b}_1}|&=\binom{K/2-1}{w-1}\binom{K/2-1}{t+1-w},\\ |\mathcal{V}_{w;\overline{a}_1,b_1}|&=\binom{K/2-1}{w}\binom{K/2-1}{t-w}, & |\mathcal{V}_{w;\overline{a}_1,\overline{b}_1}|&=\binom{K/2-1}{w}\binom{K/2-1}{t+1-w}.\end{aligned} \tag{9}$$

Let $\lambda=\frac{M}{N}$. So $\lambda=\frac{t}{K}$ in MN scheme. Given a fixed number λ, Table II can be obtained by (9) when K is appropriate large.

Now let us consider the sets in (4) and their subsets in (8). We can obtain a bipartite graph for any two different subsets. However we only interested in the bipartite graph which has at least one edge. It is not difficult to check that any two elements of a set can not form an effective pair since they have the same number of users requiring from server A(and sever B). So we only need to consider any two subsets from distinct sets. We take bipartite graph $\mathbf{G}=(\mathcal{V}_{\frac{t+1}{2};\overline{a}_1,\overline{b}_1},\mathcal{V}_{\frac{t-1}{2};\overline{a}_1,b_1};\mathcal{E})$ as an example. First let us count the degree of each vertex in $\mathcal{V}_{\frac{t+1}{2};\overline{a}_1,\overline{b}_1}$. Given a vertex

$$\mathcal{S}=\{a_{i_1},a_{i_2},\ldots,a_{i_{(t+1)/2}},b_{i'_1},b_{i'_2},\ldots,b_{i'_{(t+1)/2}}\}\in\mathcal{V}_{\frac{t+1}{2};\overline{a}_1,\overline{b}_1},$$

$1\notin\{i_1,\ldots,i_{(t+1)/2},i'_1,\ldots,i'_{(t+1)/2}\}$, it is adjacent to $(t+1)/2$ vertices

$$\mathcal{S}_j=\mathcal{S}\bigcup\{b_1\}\setminus\{a_{i_j}\}\in\mathcal{V}_{\frac{t-1}{2};\overline{a}_1,b_1}$$

TABLE II

THE CARDINALITY OF SUBSETS IN (8)

Subsets	Cardinality	$\lvert\mathcal{V}_{w;,}\rvert/\lvert\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1}\rvert$	Approximate of $\lvert\mathcal{V}_{w;,}\rvert/\lvert\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1}\rvert$
$\mathcal{V}_{\frac{t-1}{2};a_1,b_1}$	$\binom{K/2-1}{(t-3)/2}\binom{K/2-1}{(t+1)/2}$	$\frac{(t+1)(t-1)}{(K-t+1)(K-t-1)}$	$\frac{\lambda^2}{(1-\lambda)^2}$
$\mathcal{V}_{\frac{t-1}{2};a_1,\bar{b}_1}$	$\binom{K/2-1}{(t-3)/2}\binom{K/2-1}{(t+3)/2}$	$\frac{(t+1)(t-1)(K-t-3)}{(t+3)(K-t+1)(K-t-1)}$	$\frac{\lambda}{1-\lambda}$
$\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,b_1}$	$\binom{K/2-1}{(t-1)/2}\binom{K/2-1}{(t+1)/2}$	$\frac{t+1}{K-t-1}$	$\frac{\lambda}{1-\lambda}$
$\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,\bar{b}_1}$	$\binom{K/2-1}{(t-1)/2}\binom{K/2-1}{(t+3)/2}$	$\frac{(t+1)(K-t-3)}{(t+3)(K-t-1)}$	1
$\mathcal{V}_{\frac{t+1}{2};a_1,b_1}$	$\binom{K/2-1}{(t-1)/2}\binom{K/2-1}{(t-1)/2}$	$\frac{(t+1)(t+1)}{(K-t-1)(K-t-1)}$	$\frac{\lambda^2}{(1-\lambda)^2}$
$\mathcal{V}_{\frac{t+1}{2};a_1,\bar{b}_1}$	$\binom{K/2-1}{(t-1)/2}\binom{K/2-1}{(t+1)/2}$	$\frac{t+1}{K-t-1}$	$\frac{\lambda}{1-\lambda}$
$\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,b_1}$	$\binom{K/2-1}{(t+1)/2}\binom{K/2-1}{(t-1)/2}$	$\frac{t+1}{K-t-1}$	$\frac{\lambda}{1-\lambda}$
$\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1}$	$\binom{K/2-1}{(t+1)/2}\binom{K/2-1}{(t+1)/2}$	1	1
$\mathcal{V}_{\frac{t+3}{2};a_1,b_1}$	$\binom{K/2-1}{(t+1)/2}\binom{K/2-1}{(t-3)/2}$	$\frac{(t+1)(t-1)}{(K-t+1)(K-t-1)}$	$\frac{\lambda^2}{(1-\lambda)^2}$
$\mathcal{V}_{\frac{t+3}{2};a_1,\bar{b}_1}$	$\binom{K/2-1}{(t+1)/2}\binom{K/2-1}{(t-1)/2}$	$\frac{t+1}{K-t-1}$	$\frac{\lambda}{1-\lambda}$
$\mathcal{V}_{\frac{t+3}{2};\bar{a}_1,b_1}$	$\binom{K/2-1}{(t+3)/2}\binom{K/2-1}{(t-3)/2}$	$\frac{(t+1)(t-1)(K-t-3)}{(t+3)(K-t+1)(K-t-1)}$	$\frac{\lambda}{1-\lambda}$
$\mathcal{V}_{\frac{t+3}{2};\bar{a}_1,\bar{b}_1}$	$\binom{K/2-1}{(t+3)/2}\binom{K/2-1}{(t-1)/2}$	$\frac{(t+1)(K-t-3)}{(t+3)(K-t-1)}$	1

Hence $d(\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1}) = (t+1)/2$. Now let us consider the degree of each vertex in $\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,b_1}$. Given a vertex

$$\mathcal{S} = \{a_{i_1}, a_{i_2}, \ldots, a_{i_{(t-1)/2}}, b_{i'_1}, b_{i'_2}, \ldots, b_{i'_{(t+1)/2}}, b_1\} \in \mathcal{V}_{\frac{t-1}{2};\bar{a}_1,b_1},$$

$1 \notin \{i_1, \ldots, i_{(t-1)/2}, i'_1, \ldots, i'_{(t+1)/2}\}$, it is adjacent to $\frac{K}{2} - \frac{t+1}{2} = \frac{K-t-1}{2}$ vertices, i.e., $\mathcal{S} \bigcup \{a\} \setminus \{b_1\} \in \mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1}$ for each $a \in \mathcal{K}_A \setminus (\{a_1\} \bigcup \mathcal{S})$. That is $d(\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,b_1}) = \frac{K-t-1}{2}$. Similarly we can compute the degree of each vertex in the bipartite graph $\mathbf{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{E})$ generated by any two subsets from distinct sets and list them in Tables III and IV where $d(\mathcal{X})$ and $d(\mathcal{Y})$ are respectively on the the top and bottom of the diagonal in the entry indexed by $\mathcal{X}$ and $\mathcal{Y}$. It is easy to check that the elements on the top and bottom of the diagonal in the entry indexed by $(\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,b_1}, \mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1})$ are $\frac{t+1}{2}$ and $\frac{K-t-1}{2}$ respectively. By the way the entry is defined by empty when there is no edges in the related bipartite graph.

TABLE III

THE DEGREES FOR EACH BIPARTITE GRAPH (I)

$\mathcal{V}_{\frac{t+1}{2}}$ / $d(\mathcal{V}_{\frac{t+1}{2}})$ / $\mathcal{V}$ / $d(\mathcal{V})$	$\mathcal{V}_{\frac{t+1}{2},a_1,b_1}$	$\mathcal{V}_{\frac{t+1}{2},a_1,\bar{b}_1}$	$\mathcal{V}_{\frac{t+1}{2},\bar{a}_1,b_1}$	$\mathcal{V}_{\frac{t+1}{2},\bar{a}_1,\bar{b}_1}$
$\mathcal{V}_{\frac{t-1}{2},a_1,b_1}$	$\frac{(t-1)(K-t-1)}{4}$ / $\frac{(K-t+1)(t+1)}{4}$	$\frac{(t-1)(K-t-3)}{4}$ / $\frac{(K-t+1)(t+3)}{4}$		
$\mathcal{V}_{\frac{t-1}{2},a_1,\bar{b}_1}$		$\frac{(t-1)(K-t-3)}{4}$ / $\frac{(K-t+1)(t+3)}{4}$		
$\mathcal{V}_{\frac{t-1}{2},\bar{a}_1,b_1}$	$\frac{K-t-1}{2}$ / $\frac{t+1}{2}$	1 / 1	$\frac{(K-t-1)(t+1)}{4}$ / $\frac{(K-t-1)(t+1)}{4}$	$\frac{t+1}{2}$ / $\frac{K-t-1}{2}$
$\mathcal{V}_{\frac{t-1}{2},\bar{a}_1,\bar{b}_1}$		$\frac{K-t-3}{2}$ / $\frac{t+3}{2}$		$\frac{(t+1)(K-t-3)}{4}$ / $\frac{(K-t-3)(t+3)}{4}$
$\mathcal{V}_{\frac{t+3}{2},a_1,b_1}$	$\frac{(t-1)(K-t-1)}{4}$ / $\frac{(t+1)(K-t+1)}{4}$		$\frac{t-1}{2}$ / $\frac{K-t+1}{2}$	
$\mathcal{V}_{\frac{t+3}{2},a_1,\bar{b}_1}$	$\frac{K-t-1}{2}$ / $\frac{t+1}{2}$	$\frac{(K-t-1)(t+1)}{4}$ / $\frac{(t+1)(K-t-1)}{4}$	1 / 1	$\frac{t+1}{2}$ / $\frac{K-t-1}{2}$
$\mathcal{V}_{\frac{t+3}{2},\bar{a}_1,b_1}$			$\frac{(K-t-3)(t-1)}{4}$ / $\frac{(t+3)(K-t+1)}{4}$	
$\mathcal{V}_{\frac{t+3}{2},\bar{a}_1,\bar{b}_1}$			$\frac{K-t-3}{2}$ / $\frac{t+3}{2}$	$\frac{(K-t-3)(t+1)}{4}$ / $\frac{(t+3)(K-t-1)}{4}$

In order to make the cardinality of a maximal matching as large as possible, we can also use several subsets to generate a bipartite graph.

Example 1: A bipartite graph $\mathbf{G} = (\mathcal{X}, \mathcal{Y}_2; \mathcal{E})$ where

$$\mathcal{X} = \mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1}, \qquad \mathcal{Y} = \mathcal{V}_{\frac{t-1}{2};\bar{a}_1,b_1} \bigcup \mathcal{V}_{\frac{t+3}{2};a_1,\bar{b}_1}, \tag{10}$$

TABLE IV
THE DEGREES FOR EACH BIPARTITE GRAPH (II)

$\mathcal{V}_{\frac{t+3}{2}}$ / $d(\mathcal{V}_{\frac{t+3}{2}})$ / $\mathcal{V}_{\frac{t-1}{2}}$ / $d(\mathcal{V}_{\frac{t-1}{2}})$	$\mathcal{V}_{\frac{t+3}{2},a_1,b_1}$	$\mathcal{V}_{\frac{t+3}{2},a_1,\bar{b}_1}$	$\mathcal{V}_{\frac{t+3}{2},\bar{a}_1,b_1}$	$\mathcal{V}_{\frac{t+3}{2},\bar{a}_1,\bar{b}_1}$
$\mathcal{V}_{\frac{t-1}{2},a_1,b_1}$	$\binom{\frac{t+1}{2}}{2}\binom{\frac{K-t+1}{2}}{2}$ / $\binom{\frac{K-t+1}{2}}{2}\binom{\frac{t+1}{2}}{2}$	$\frac{K-t-1}{2}\binom{\frac{t+1}{2}}{2}$ / $\frac{t+1}{2}\binom{\frac{K-t+1}{2}}{2}$		
$\mathcal{V}_{\frac{t-1}{2},a_1,\bar{b}_1}$		$\binom{\frac{t+1}{2}}{2}\binom{\frac{K-t-1}{2}}{2}$ / $\binom{\frac{K-t+1}{2}}{2}\binom{\frac{t+3}{2}}{2}$		
$\mathcal{V}_{\frac{t-1}{2},\bar{a}_1,b_1}$	$\frac{t+1}{2}\binom{\frac{K-t+1}{2}}{2}$ / $\frac{K-t-1}{2}\binom{\frac{t+1}{2}}{2}$	$\frac{(t+1)(K-t+1)}{4}$ / $\frac{(K-t+1)(t+1)}{4}$	$\binom{\frac{t+3}{2}}{2}\binom{\frac{K-t+1}{2}}{2}$ / $\binom{\frac{K-t-1}{2}}{2}\binom{\frac{t+1}{2}}{2}$	$\frac{K-t-1}{2}\binom{\frac{t+3}{2}}{2}$ / $\frac{t+1}{2}\binom{\frac{K-t-1}{2}}{2}$
$\mathcal{V}_{\frac{t-1}{2},\bar{a}_1,\bar{b}_1}$		$\frac{t+1}{2}\binom{\frac{K-t-1}{2}}{2}$ / $\frac{K-t-1}{2}\binom{\frac{t+3}{2}}{2}$		$\binom{\frac{t+3}{2}}{2}\binom{\frac{K-t-1}{2}}{2}$ / $\binom{\frac{K-t-1}{2}}{2}\binom{\frac{t+3}{2}}{2}$

can be obtained. From Table III, we have $d(\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1}) = \frac{t+1}{2} + \frac{t+1}{2} = t+1$ and $d(\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,b_1} \bigcup \mathcal{V}_{\frac{t+3}{2};a_1,\bar{b}_1}) = \frac{K-t+1}{2}$. Suppose that $\lambda \in (0, \frac{3-\sqrt{5}}{2})$. It is easy to check that $t+1 \leq \frac{K-t+1}{2}$ if $\lambda \in (0, \frac{1}{3} - \frac{1}{K}]$, otherwise $t+1 \geq \frac{K-t+1}{2}$. From Corollary 1 there is a saturating matching. So there are $|\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,b_1} \bigcup \mathcal{V}_{\frac{t+3}{2};a_1,\bar{b}_1}|$ or $|\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1}|$ vertices in the maximal matching of $\mathbf{G}$ generated by (10). Of course we can also assume that

$$\mathcal{X} = \mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1}, \quad \mathcal{Y} = \mathcal{V}_{\frac{t-1}{2};\bar{a}_1,b_1} \qquad \text{or} \qquad \mathcal{X} = \mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1}, \quad \mathcal{Y} = \mathcal{V}_{\frac{t+3}{2};a_1,\bar{b}_1}.$$

Similarly we can also show that they have saturating matchings respectively. And it is easy to check that cardinality of the maximal matching by the first assumption is maximal.

With the aid of a computer, we have the following bipartite graphes such that the the number of the unpair of messages is minimal according to the value of λ.

A. $0 < \lambda \leq \frac{3-\sqrt{5}}{2}$

When $\lambda \leq \frac{3-\sqrt{5}}{2}$, one of the most appropriate method constructing bipartite graphs is

$$\begin{aligned}
&\mathbf{G}_1 = (\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1}, \mathcal{V}_{\frac{t-1}{2};\bar{a}_1,b_1} \bigcup \mathcal{V}_{\frac{t+3}{2};a_1,\bar{b}_1}; \mathcal{E}_1) \\
&\mathbf{G}_2 = (\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,b_1}, \mathcal{V}_{\frac{t+3}{2};\bar{a}_1,b_1}; \mathcal{E}_2) \qquad \mathbf{G}_3 = (\mathcal{V}_{\frac{t+1}{2};a_1,\bar{b}_1}, \mathcal{V}_{\frac{t-1}{2};a_1,\bar{b}_1}; \mathcal{E}_3) \\
&\mathbf{G}_4 = (\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,\bar{b}_1}, \mathcal{V}_{\frac{t+3}{2};\bar{a}_1,\bar{b}_1}; \mathcal{E}_4) \qquad \mathbf{G}_5 = (\mathcal{V}_{\frac{t-1}{2};a_1,b_1}, \mathcal{V}_{\frac{t+3}{2};a_1,b_1}; \mathcal{E}_5)
\end{aligned} \tag{11}$$

From Tables III and IV, similar to the discussion in Example 1 the following statement holds.

Lemma 1: Each of the bipartite graphs in (11) has a saturating matching.

From Lemma 1 and Table II, the number of unpaired messages is

$$\begin{aligned}
n_1 = &\left|\mathcal{V}_{\frac{t+1}{2};a_1,b_1}\right| + \left|\left|\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1}\right| - \left|\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,b_1} \bigcup \mathcal{V}_{\frac{t+3}{2};a_1,\bar{b}_1}\right|\right| + \left|\left|\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,b_1}\right| - \left|\mathcal{V}_{\frac{t+3}{2};\bar{a}_1,b_1}\right|\right| \\
&+ \left|\left|\mathcal{V}_{\frac{t+1}{2};a_1,\bar{b}_1}\right| - \left|\mathcal{V}_{\frac{t-1}{2};a_1,\bar{b}_1}\right|\right| + \left|\left|\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,\bar{b}_1}\right| - \left|\mathcal{V}_{\frac{t+3}{2};\bar{a}_1,,\bar{b}_1}\right|\right| + \left|\left|\mathcal{V}_{\frac{t-1}{2};a_1,b_1}\right| - \left|\mathcal{V}_{\frac{t+3}{2};a_1,b_1}\right|\right| \\
&= \left[\left(\frac{t+1}{K-t-1}\right)^2 + \left|1 - 2\frac{t+1}{K-t-1}\right| + 2\left|\frac{t+1}{K-t-1} - \frac{(t+1)(t-1)(K-t-3)}{(t+3)(K-t+1)(K-t-1)}\right|\right] \binom{K/2-1}{(t+1)/2}^2 \\
&= \left[\frac{|K-3t-3|}{K-t-1} + \frac{(t+1)^2(K-t+1)(t+3)+8K(t+1)(K-t-1)}{(K-t-1)^2(t+3)(K-t+1)}\right] \binom{K/2-1}{(t+1)/2}^2.
\end{aligned}$$

Now let us consider the reduction comparing with the scheme in [5], i.e.,

$$\begin{aligned}\frac{n_1}{n} &= \frac{\left[\frac{|K-3t-3|}{K-t-1} + \frac{(t+1)^2(K-t+1)(t+3)+8K(t+1)(K-t-1)}{(K-t-1)^2(t+3)(K-t+1)}\right]\binom{K/2-1}{(t+1)/2}^2}{\left|\binom{K_A}{(t+1)/2}\binom{K_B}{(t+1)/2} - \binom{K_A}{(t-1)/2}\binom{K_B}{(t+3)/2} - \binom{K_B}{(t-1)/2}\binom{K_A}{(t+3)/2}\right|} \\ &= \left(\frac{|K-3t-3|}{K-t-1} + \frac{(t+1)^2(K-t+1)(t+3)+8K(t+1)(K-t-1)}{(K-t-1)^2(t+3)(K-t+1)}\right)\frac{\frac{(K-t-1)^2}{K^2}}{2\frac{(t+1)(K-t-1)}{(K-t+1)(t+3)} - 1} \\ &\approx |1-3\lambda|(1-\lambda) + \lambda^2.\end{aligned} \tag{12}$$

The last equation holds when K is appropriate large and λ is a fixed number. This implies that the number of unpaired messages left by (11) is about $\frac{|1-3\lambda|(1-\lambda)+\lambda^2}{3}$ times smaller than that of obtained by (5). Figure 1 is the function n_1/n depends on variable $\lambda \in (0, \frac{3-\sqrt{5}}{2}]$. Clearly $n_1/n \approx \frac{1}{9}$ if λ towards $\frac{1}{3}$. We can also compute the ratio of unpaired messages as follows.

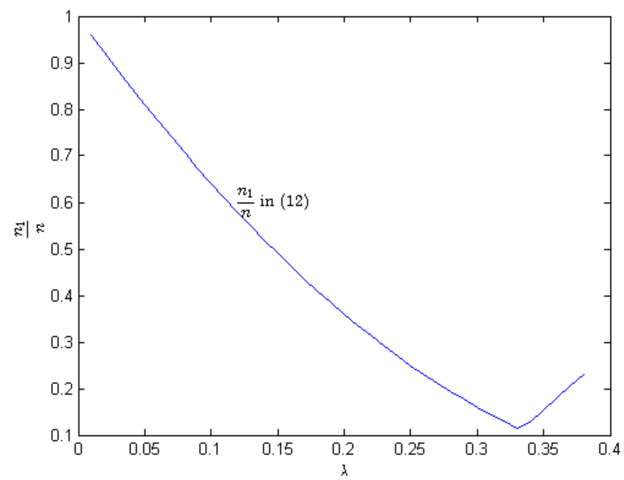

Fig. 1. The function $\frac{n_1}{n}$ in (12) depends on variable $0 < \lambda \leq 3 - \sqrt{5})/2$

$$\begin{aligned}\Delta_1(\lambda) &= \frac{n_1}{\binom{K}{t+1}} < \frac{n_1}{\left|\mathcal{V}_{\frac{t+1}{2}}\right| + \left|\mathcal{V}_{\frac{t-1}{2}} \bigcup \mathcal{V}_{\frac{t+3}{2}}\right|} \\ &= \left[\frac{|K-3t-3|}{K-t-1} + \frac{(t+1)^2(K-t+1)(t+3)+8K(t+1)(K-t+1)}{(K-t-1)^2(t+3)(K-t+1)}\right]\frac{\binom{K/2-1}{(t+1)/2}^2}{2\binom{K/2}{(t-1)/2}\binom{K/2}{(t+3)/2}+\binom{K/2}{(t+1)/2}\binom{K/2}{(t+1)/2}} \\ &= \left[\frac{|K-3t-3|}{K-t-1} + \frac{(t+1)^2(K-t+1)(t+3)+8K(t+1)(K-t-1)}{(K-t-1)^2(t+3)(K-t+1)}\right]\frac{\frac{(K-t-1)^2}{K^2}}{2\frac{(t+1)(K-t-1)}{(K-t+1)(t+3)} + 1} \\ &\approx \frac{|1-3\lambda|(1-\lambda)+\lambda^2}{3}\end{aligned} \tag{13}$$

Clearly $\Delta_1(\lambda)$ tends to $\frac{1}{27}$ if λ towards $\frac{1}{3}$.

B. $\frac{3-\sqrt{5}}{2} < \lambda \leq \frac{\sqrt{5}-1}{2}$

When $\frac{3-\sqrt{5}}{2} < \lambda \leq \frac{\sqrt{5}-1}{2}$, one of the most appropriate method constructing bipartite graphs is

$$\begin{aligned}&\mathbf{G}_1 = (\mathcal{V}_{\frac{t+1}{2};a_1,b_1}, \mathcal{V}_{\frac{t-1}{2};\overline{a}_1,b_1}; \mathcal{E}_1), && \mathbf{G}_2 = (\mathcal{V}_{\frac{t+1}{2};a_1,\overline{b}_1}, \mathcal{V}_{\frac{t-1}{2};a_1,\overline{b}_1}; \mathcal{E}_2) \\ &\mathbf{G}_3 = (\mathcal{V}_{\frac{t+1}{2};\overline{a}_1,b_1}, \mathcal{V}_{\frac{t+3}{2};\overline{a}_1,b_1}; \mathcal{E}_3) && \mathbf{G}_4 = (\mathcal{V}_{\frac{t+1}{2};\overline{a}_1,\overline{b}_1}, \mathcal{V}_{\frac{t+3}{2};a_1,\overline{b}_1}; \mathcal{E}_4) \\ &\mathbf{G}_5 = (\mathcal{V}_{\frac{t-1}{2};a_1,b_1}, \mathcal{V}_{\frac{t+3}{2};a_1,b_1}; \mathcal{E}_5) && \mathbf{G}_6 = (\mathcal{V}_{\frac{t-1}{2};\overline{a}_1,\overline{b}_1}, \mathcal{V}_{\frac{t+3}{2};\overline{a}_1,\overline{b}_1}; \mathcal{E}_6)\end{aligned} \tag{14}$$

From Tables III and IV, similar to the discussion in Example 1 the following result can be obtained.

Lemma 2: Each of the bipartite graphs in (14) has a saturating matching.

From Lemma 2 and Table II, the number of unpaired messages is

$$\begin{aligned} n_2 &= \left|\left|\mathcal{V}_{\frac{t+1}{2};a_1,b_1}\right| - \left|\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,b_1}\right|\right| + \left|\left|\mathcal{V}_{\frac{t+1}{2};a_1,\bar{b}_1}\right| - \left|\mathcal{V}_{\frac{t-1}{2};a_1,\bar{b}_1}\right|\right| + \left|\left|\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,b_1}\right| - \left|\mathcal{V}_{\frac{t+3}{2};\bar{a}_1,b_1}\right|\right| \\ &\quad + \left|\left|\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1}\right| - \left|\mathcal{V}_{\frac{t+3}{2};a_1,\bar{b}_1}\right|\right| + \left|\left|\mathcal{V}_{\frac{t-1}{2};a_1,b_1}\right| - \left|\mathcal{V}_{\frac{t+3}{2};a_1,b_1}\right|\right| + \left|\left|\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,\bar{b}_1}\right| - \left|\mathcal{V}_{\frac{t+3}{2};\bar{a}_1,\bar{b}_1}\right|\right|. \\ &= \left(\frac{t+1}{K-t-1}\left|\frac{t+1}{K-t-1}-1\right| + 2\left|\frac{t+1}{K-t-1} - \frac{(t+1)(t-1)(K-t-3)}{(t+3)(K-t+1)(K-t-1)}\right| + \left|\frac{t+1}{K-t-1}-1\right|\right)\binom{K/2-1}{(t+1)/2}^2 \\ &= \left(\frac{K}{(K-t-1)^2}|2t+2-K| + \frac{t+1}{K-t-1}\frac{8K}{(t+3)(K-t+1)}\right)\binom{K/2-1}{(t+1)/2}^2 \end{aligned}$$

Similar to (12) and (13), we have

$$\begin{aligned} \frac{n_2}{n} &= \left(\frac{K}{(K-t-1)^2}|2t+2-K| + \frac{t+1}{K-t-1}\frac{8K}{(t+3)(K-t+1)}\right)\frac{\frac{(K-t-1)^2}{K^2}}{2\frac{(t+1)(K-t-1)}{(K-t+1)(t+3)}-1} \\ &\approx |2\lambda - 1| \end{aligned} \tag{15}$$

and the ratio of unpaired messages

$$\begin{aligned} \Delta_2(\lambda) = \frac{n_2}{\binom{K}{t+1}} &< \frac{n_2}{2\binom{K/2}{(t-1)/2}\binom{K/2}{(t+3)/2} + \binom{K/2}{(t+1)/2}\binom{K/2}{(t+1)/2}} \\ &\approx \frac{1}{3}|2\lambda - 1| \end{aligned} \tag{16}$$

Clearly both n_2/n and $\Delta_2(\lambda)$ tend to 0 if λ towards $1/2$. Figure 2 is the function n_2/n depends on variable $\lambda \in (\frac{3-\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}]$.

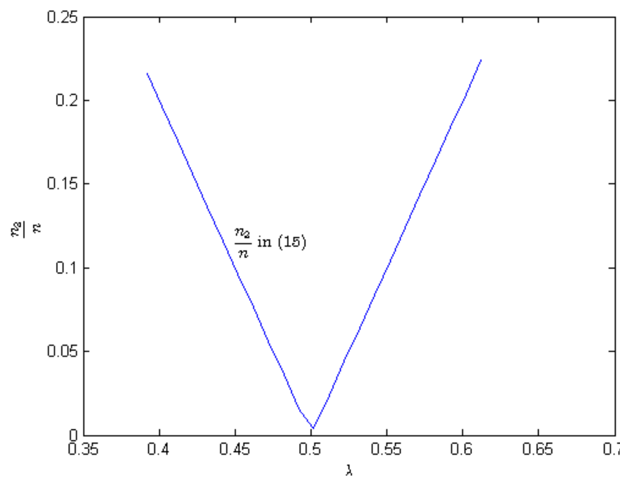

Fig. 2. The function $\frac{n_2}{n}$ in (15) depends on variable $\frac{3-\sqrt{5}}{2} < \lambda \leq \frac{\sqrt{5}-1}{2}$

C. $\frac{\sqrt{5}-1}{2} < \lambda < 1$

When $\frac{\sqrt{5}-1}{2} < \lambda < 1$, one of the most appropriate method constructing bipartite graphs is

$$\begin{aligned} &\mathbf{G}_1 = (\mathcal{V}_{\frac{t+1}{2};a_1,b_1}, \mathcal{V}_{\frac{t-1}{2};\bar{a}_1,b_1}\bigcup \mathcal{V}_{\frac{t+3}{2};a_1,\bar{b}_1}; \mathcal{E}_1) \\ &\mathbf{G}_2 = (\mathcal{V}_{\frac{t+1}{2};a_1,\bar{b}_1}, \mathcal{V}_{\frac{t-1}{2};a_1,\bar{b}_1}; \mathcal{E}_2) \qquad \mathbf{G}_3 = (\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,b_1}, \mathcal{V}_{\frac{t+3}{2};\bar{a}_1,b_1}; \mathcal{E}_3) \\ &\mathbf{G}_4 = (\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,\bar{b}_1}, \mathcal{V}_{\frac{t+3}{2};\bar{a}_1,\bar{b}_1}; \mathcal{E}_4) \qquad \mathbf{G}_5 = (\mathcal{V}_{\frac{t-1}{2};a_1,b_1}, \mathcal{V}_{\frac{t+3}{2};a_1,b_1}; \mathcal{E}_5) \end{aligned} \tag{17}$$

Similar to the discussions in Section III-A, the following results can be obtained.

Lemma 3: Each of the bipartite graphs in (17) has a saturating matching.

From Lemma 3 and Table II, the number of unpaired messages is

$$\begin{aligned} n_3 &= \left|\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,\bar{b}_1}\right| + \left|\left|\mathcal{V}_{\frac{t+1}{2};a_1,b_1}\right| - \left|\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,b_1}\bigcup\mathcal{V}_{\frac{t+3}{2};a_1,\bar{b}_1}\right|\right| + \left|\left|\mathcal{V}_{\frac{t+1}{2};a_1,\bar{b}_1}\right| - \left|\mathcal{V}_{\frac{t-1}{2};a_1,\bar{b}_1}\right|\right| \\ &+ \left|\left|\mathcal{V}_{\frac{t+1}{2};\bar{a}_1,b_1}\right| - \left|\mathcal{V}_{\frac{t+3}{2};\bar{a}_1,b_1}\right|\right| + \left|\left|\mathcal{V}_{\frac{t-1}{2};\bar{a}_1,\bar{b}_1}\right| - \left|\mathcal{V}_{\frac{t+3}{2};\bar{a}_1,,\bar{b}_1}\right|\right| + \left|\left|\mathcal{V}_{\frac{t-1}{2};a_1,b_1}\right| - \left|\mathcal{V}_{\frac{t+3}{2};a_1,b_1}\right|\right| \\ &= \left(1+\tfrac{t+1}{K-t-1}\left|\tfrac{t+1}{K-t-1}-2\right| + 2\left|\tfrac{t+1}{K-t-1} - \tfrac{(t+1)(t-1)(K-t-3)}{(t+3)(K-t+1)(K-t-1)}\right|\right)\binom{K/2-1}{(t+1)/2}^2 \\ &= \left(1+\tfrac{K}{(K-t-1)^2}|3t+3-2K| + \tfrac{t+1}{K-t-1}\tfrac{8K}{(t+3)(K-t+1)}\right)\binom{K/2-1}{(t+1)/2}^2 \end{aligned}$$

Similar to (12) and (13), we have

$$\begin{aligned} \frac{n_3}{n} &= \left(1+\tfrac{K}{(K-t-1)^2}|3t+3-2K| + \tfrac{t+1}{K-t-1}\tfrac{8K}{(t+3)(K-t+1)}\right)\frac{\frac{(K-t-1)^2}{K^2}}{2\frac{(t+1)(K-t-1)}{(K-t+1)(t+3)}-1} \\ &\approx (1-\lambda)^2 + \lambda\,|3\lambda-2| \end{aligned} \tag{18}$$

and the ratio of unpaired messages left

$$\begin{aligned} \Delta_3(\lambda) = \frac{n_3}{\binom{K}{t+1}} &< \frac{n_3}{2\binom{K/2}{(t-1)/2}\binom{K/2}{(t+3)/2}+\binom{K/2}{(t+1)/2}\binom{K/2}{(t+1)/2}} \\ &= \left[1+\tfrac{K}{K-t-1}\left|\tfrac{t+1}{K-t-1}-2\right| + \tfrac{t+1}{K-t-1}\tfrac{8K}{(t+3)(K-t+1)}\right]\frac{\binom{K/2-1}{(t+1)/2}^2}{2\binom{K/2}{(t-1)/2}\binom{K/2}{(t+3)/2}+\binom{K/2}{(t+1)/2}\binom{K/2}{(t+1)/2}} \\ &\approx \tfrac{(1-\lambda)^2+\lambda|3\lambda-2|}{3} \end{aligned} \tag{19}$$

Clearly n_3/n and $\Delta_3(\lambda)$ tend to $\frac{1}{9}$ and $\frac{1}{27}$ respectively if λ towards $\frac{2}{3}$. Figure 3 is the function n_3/n depends on variable $\lambda \in (\frac{\sqrt{5}-1}{2}, 1)$.

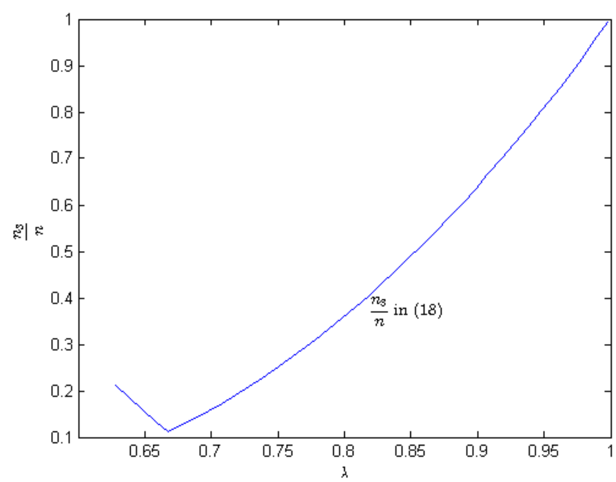

Fig. 3. The function $\frac{n_3}{n}$ in (18) depends on variable $\frac{\sqrt{5}-1}{2} < \lambda < 1$

Theorem 3: Based on MN scheme, when $K_A = K_B$, for the server system in Table I the rate is

$$R_T(K, \tfrac{M}{N}) = \begin{cases} \frac{1}{2}R_{MN}(K, \frac{M}{N}) & \text{if } K\frac{M}{N} \text{ is even} \\ (\frac{1}{2}+\frac{1}{6}\Delta'(\frac{M}{N}))R_{MN}(K, \frac{M}{N}) & \text{if } K\frac{M}{N} \text{ is odd} \end{cases} \tag{20}$$

where Δ', which is the minimum value of $\Delta(\frac{M}{N})$ in (7), $\Delta_1(\frac{M}{N})$ in (13), $\Delta_2(\frac{M}{N})$ in (16) and $\Delta_3(\frac{M}{N})$ in (19), represents the ratio of unpaired messages.

Finally, we should point out that the rate in Theorem 3 can be further improved when t is odd. First let us generalize the notations in (8), i.e., define

$$\mathcal{V}_{w;\overline{a}_1,\ldots,\overline{a}_{h_1-1},a_{h_1},\overline{b}_1,\ldots,\overline{b}_{h_2-1},b_{h_2}} = \{\mathcal{S} \in \mathcal{V}_w \mid a_{h_1}, b_{h_2} \in \mathcal{S}, a_i, b_j \notin \mathcal{S}, i \in [1, h_1), j \in [1, h_2)\} \tag{21}$$

where $w = \frac{t-1}{2}$, $\frac{t+1}{2}$, $\frac{t+3}{2}$ and $h_1 \in [1, \frac{K}{2} - w + 1]$, $h_2 \in [1, \frac{K}{2} - t + w]$. It is easy to check that

$$|\mathcal{V}_{w;\overline{a}_1,\ldots,\overline{a}_{h_1-1},a_{h_1},\overline{b}_1,\ldots,\overline{b}_{h_2-1},b_{h_2}}| = \binom{K/2-h_1}{w-1}\binom{K/2-h_2}{t-w} \tag{22}$$

and

$$\mathcal{V}_{w;\overline{a}_1,\ldots,\overline{a}_{h_1-1},a_{h_1},\overline{b}_1,\ldots,\overline{b}_{h_2-1},b_{h_2}} \bigcap \mathcal{V}_{w;\overline{a}_1,\ldots,\overline{a}_{h_1'-1},a_{h_1'},\overline{b}_1,\ldots,\overline{b}_{h_2'-1},b_{h_2'}} = \emptyset$$

for any distinct vectors $(h_1, h_2) \neq (h_1', h_2')$. In addition,

$$\binom{K/2}{w}\binom{K/2}{t+1-w} = \sum_{h_1=1}^{K/2-w+1} \sum_{h_2=1}^{K/2-t+w} \binom{K/2-h_1}{w-1}\binom{K/2-h_2}{t-w} \tag{23}$$

since it is well know

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-2}{m-1} + \ldots + \binom{m-1}{m-1}, \qquad 1 \leq m < n$$

always holds. So we have

$$\mathcal{V}_w = \bigcup_{h_1=1}^{K/2-w+1} \bigcup_{h_2=1}^{K/2-t+w} \mathcal{V}_{w;\overline{a}_1,\ldots,\overline{a}_{h_1-1},a_{h_1},\overline{b}_1,\ldots,\overline{b}_{h_2-1},b_{h_2}}.$$

Then we can also compute the degree of each vertex in the bipartite graph generated by any subsets in (21). Similar to the discussions in Subsections III-A, III-B and III-C, we can further reduce the value of $\Delta'(\frac{M}{N})$ in (20) by sacrificing run-time efficiency on constructing the most appropriate classes of bipartite graphs. In fact the sacrificing run-time is very small comparing with that of finding the maximal matching of graph $\mathbf{G} = (\mathcal{V}_{\frac{t-1}{2}} \bigcup \mathcal{V}_{\frac{t+1}{2}} \bigcup \mathcal{V}_{\frac{t+3}{2}}; \mathcal{E})$.

IV. Conclusion

In this paper, we modified the scheme for multi-servers setting in [5] when $K\frac{M}{N}$ is odd. Consequently an obviously smaller rate was obtained. Especially when K is large, $R \approx \frac{1}{2}R_{MN}(K, \frac{M}{N})$ if $\frac{M}{N}$ nears $1/2$, and $R \approx \frac{41}{81}R_{MN}(K, \frac{M}{N})$ if $\frac{M}{N}$ nears $\frac{1}{3}$ or $\frac{2}{3}$. In addition, our modification can be generalized to further reduce the rate. However with an exhaustive computer search, it will cost more running times to search the bipartite graphs generated by the subsets in (21) such that the unpaired messages as small as possible. So it would be of interest if we can propose a determined construction of such bipartite graphs.

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