

MIS in the Congested Clique Model in $O(\log \log \Delta)$ Rounds*

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Abstract. We give a maximal independent set (MIS) algorithm that runs in $O(\log \log \Delta)$ rounds in the congested clique model, where Δ is the maximum degree of the input graph. This improves upon the $O(\frac{\log(\Delta) \cdot \log \log \Delta}{\sqrt{\log n}} + \log \log \Delta)$ rounds algorithm of [Ghaffari, PODC '17], where n is the number of vertices of the input graph.

In the first stage of our algorithm, we simulate the first $O(\frac{n}{\text{poly} \log n})$ iterations of the sequential random order Greedy algorithm for MIS in the congested clique model in $O(\log \log \Delta)$ rounds. This thins out the input graph relatively quickly: After this stage, the maximum degree of the residual graph is poly-logarithmic. In the second stage, we run the MIS algorithm of [Ghaffari, PODC '17] on the residual graph, which completes in $O(\log \log \Delta)$ rounds on graphs of poly-logarithmic degree.

1 Introduction

The LOCAL and CONGEST Models. The LOCAL [19,23] and CONGEST [23] models are the most studied computational models for distributed graph algorithms. In these models, a communication network is represented by an n -vertex graph $G = (V, E)$, which also constitutes the input to a computational graph problem. Each vertex (or network node) $v \in V$ hosts a computational unit and is identified by a unique ID $\in \Theta(\log n)$. Initially, besides its ID, every vertex knows its neighbors (and their IDs). All network nodes simultaneously commence the execution of a distributed algorithm. Such an algorithm proceeds in synchronous rounds, where each round consists of two phases. In the computation phase, every vertex may execute unlimited computations. This is followed by the communication phase, where vertices may exchange individual messages with their neighbors. While message lengths are unbounded in the LOCAL model, in the CONGEST model every message is of length $O(\log n)$. The goal is to design algorithms that employ as few communication rounds as possible. The output is typically distributed. For independent set problems, which are the focus of this paper, upon termination of the algorithm, every vertex knows whether it participates in the independent set.

The LOCAL model provides an abstraction that allows for the study of the *locality* of a distributed problem, i.e., how far network nodes need to be able to look into the network in order to complete a certain task. In addition to the locality constraint, the CONGEST model also addresses the issue of *congestion*. For example, while in the LOCAL model, network nodes can learn their distance- r neighborhoods in r rounds, this is generally not possible in the CONGEST model due to the limitation of message sizes.

The CONGESTED-CLIQUE Model. In recent years, the CONGESTED-CLIQUE model [20], a variant of the CONGEST model, has received significant attention (e.g. [13,8,14,9,17,4,12,5,11,18]). It differs from the CONGEST model in that every pair of vertices (as opposed to only every pair of adjacent vertices) can exchange messages of sizes $O(\log n)$ in the communication phase. The focus of this model thus solely lies on the issue of congestion, since non-local message exchanges are now possible. This model is at least as powerful as the CONGEST model, and many problems, such as computing a minimum spanning tree [13,9] or computing the size of a maximum

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matching [17], can in fact be solved much faster than in the CONGEST model. In [8], Ghaffari asks whether any of the classic local problems - maximal independent set (MIS), maximal matching, $(\Delta + 1)$ -vertex-coloring, and $(2\Delta - 1)$ -edge-coloring - can be solved much faster in the CONGESTED-CLIQUE model than in the CONGEST model, where Δ is the maximum degree of the input graph. Ghaffari made progress on this question and gave a $O(\frac{\log(\Delta) \cdot \log \log \Delta}{\sqrt{\log n}} + \log \log \Delta)$ rounds MIS algorithm in the CONGESTED-CLIQUE model, while the best known CONGEST model algorithm runs in $O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ rounds [7]. This algorithm separates the two models with regards to the MIS problem, since it is known that $\Omega(\min\{\frac{\log \Delta}{\log \log \Delta}, \sqrt{\frac{\log n}{\log \log n}}\})$ rounds are required for MIS in the CONGEST model [16,15]¹.

Result. While Ghaffari gave a roughly quadratic improvement over the best CONGEST model MIS algorithm, in this paper, we show that an exponential improvement is possible. Our main result is as follows:

Theorem 1 (Main Result). *Let $G = (V, E)$ be a graph with maximum degree Δ . There is a randomized algorithm in the CONGESTED-CLIQUE model that operates in (deterministic) $O(\log \log \Delta)$ rounds and outputs a maximal independent set in G with high probability.*

Techniques. Ghaffari gave a variant of his MIS algorithm that runs in $O(\log \log \Delta)$ rounds on graphs G with poly-logarithmic maximum degree, i.e., $\Delta(G) = O(\text{polylog } n)$ (Lemma 2.15. in [8])². To achieve a runtime of $O(\log \log \Delta)$ rounds even on graphs with arbitrarily large maximum degree, we give a $O(\log \log \Delta)$ rounds algorithm that computes an independent set I such that the residual graph $G \setminus I_G[I]$ ($I_G[I]$ denotes the inclusive neighborhood of I in G) has poly-logarithmic maximum degree. We then run Ghaffari’s algorithm on the residual graph to complete the independent set computation.

Our algorithm is an implementation of the sequential GREEDY algorithm for MIS in the CONGESTED-CLIQUE model. GREEDY processes the vertices of the input graph in arbitrary order and adds the current vertex to an initially empty independent set if non of its neighbors have previously been added. The key idea is to simulate multiple iterations of GREEDY in $O(1)$ rounds in the CONGESTED-CLIQUE model. A simulation of \sqrt{n} iterations in $O(1)$ rounds can be done as follows: Let $v_1 v_2 \dots v_n$ be an arbitrary ordering of the vertices (e.g. by their IDs). Observe that the subgraph $G[\{v_1, \dots, v_{\sqrt{n}}\}]$ induced by the first \sqrt{n} vertices has at most n edges. Lenzen gave a routing protocol that can be used to collect these n edges at one distinguished vertex u in $O(1)$ rounds. Vertex u then simulates the first \sqrt{n} iterations of GREEDY locally (observe that the knowledge of $G[\{v_1, \dots, v_{\sqrt{n}}\}]$ is sufficient to do this) and then notifies the nodes chosen into the independent set about their selection.

The presented simulation can be used to obtain a $O(\sqrt{n})$ rounds MIS algorithm in the CONGESTED-CLIQUE model. To reduce the number of rounds to $O(\log \log n)$, we identify a *residual sparsity* property of the GREEDY algorithm: If GREEDY processes the vertices in *uniform random order*, then the maximum degree of the residual graph after having processed the k th vertex is $O(\frac{n}{k} \log n)$ with high probability (**Lemma 1**). To make use of this property, we will thus first compute a uniform random ordering of the vertices. Then, after having processed the first \sqrt{n} vertices as above, the maximum degree in the residual graph is $\tilde{O}(\sqrt{n})^3$. This allows us to increase the block size and simulate the next $\tilde{O}(n^{3/4})$ iterations in $O(1)$ rounds: Using the fact that the maximum degree in the residual graph is $\tilde{O}(\sqrt{n})$, it is not hard to see that

¹ This lower bound even holds in the LOCAL model.

² This variant works in fact on graphs with maximum degree bounded by $2^{c\sqrt{\log n}}$, for a sufficiently small constant c , but a poly-logarithmic degree bound is sufficient for our purposes.

³ We use the notation $\tilde{O}(\cdot)$, which equals the usual $O(\cdot)$ notation where all poly-logarithmic factors are ignored.

the subgraph induced by the next $\tilde{\Theta}(n^{\frac{3}{4}})$ random vertices has a maximum degree of $\tilde{O}(n^{1/4})$ with high probability (and thus contains $O(n)$ edges). Pursuing this approach further, we can process $\tilde{\Theta}(n^{1-\frac{1}{2^i}})$ vertices in the i th block, since, by the residual sparsity lemma, the maximum degree in the i th residual graph is $\tilde{O}(n^{\frac{1}{2^i}})$. Hence, after having processed $O(\log \log n)$ blocks, the maximum degree becomes poly-logarithmic. In Section 4, we give slightly more involved arguments that show that $O(\log \log \Delta)$ iterations (as opposed to $O(\log \log n)$ iterations) are in fact enough.

The Residual Sparsity Property of GREEDY. The author is not aware of any work that exploits or mentions the residual sparsity property of the random order GREEDY algorithm for MIS. In the context of correlation clustering in the data streaming model, a similar property of a Greedy clustering algorithm was used in [1] (Lemma 19). Their lemma is in fact strong enough and can give the version required in this paper. Since [1] does not provide a proof, and the residual sparsity property is central to the functioning of our algorithm, we give a proof that follows the main idea of [1]⁴ adapted to our needs.

Further Related Work. The maximal independent set problem is one of the classic symmetry breaking problems in distributed computing. Without all-to-all communication, Luby [22] and independently Alon et al. [2] gave $O(\log n)$ rounds distributed algorithms more than 30 years ago. Barenboim et al. [3] improved on this for certain ranges of Δ and gave a $O(\log^2 \Delta) + 2^{O(\sqrt{\log \log n})}$ rounds algorithm. The currently fastest algorithm is by Ghaffari [7] and runs in $O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ rounds.

The only MIS algorithm designed in the CONGESTED-CLIQUE model is the previously mentioned algorithm by Ghaffari [8]. Ghaffari shows how multiple rounds of a CONGEST model algorithm can be simulated in much fewer rounds in the CONGESTED-CLIQUE model. This is similar to the approach taken in this paper, however, while in our algorithm the simulation of multiple iterations of the sequential GREEDY algorithm is performed at one distinguished node, every node participates in the simulation of the CONGEST model algorithm in Ghaffari’s algorithm.

Outline. We proceed as follows. First, we give necessary definitions and notation, and we state known results that we employ in this paper (Section 2). We then give a proof of the residual sparsity property of the sequential GREEDY algorithm (Section 3). Our $O(\log \log \Delta)$ rounds MIS algorithm is subsequently presented (Section 4), followed by a brief conclusion (Section 5).

2 Preliminaries

We assume that $G = (V, E)$ is a simple unweighted n -vertex graph. For a node $v \in V$, we write $\Gamma_G(v)$ to denote v ’s (exclusive) neighborhood, and we write $\deg_G(v) := |\Gamma_G(v)|$. The inclusive neighborhood is defined as $\Gamma_G[v] := \Gamma_G(v) \cup \{v\}$. Inclusive neighborhoods are extended to subsets $U \subseteq V$ as $\Gamma_G[U] := \cup_{u \in U} \Gamma_G[u]$. Given a subset of vertices $U \subseteq V$, the subgraph induced by U is denoted by $G[U]$.

Independent Sets. An *independent set* $I \subseteq V$ is a subset of non-adjacent vertices. An independent set I is *maximal* if for every $v \in V \setminus I$, $I \cup \{v\}$ is not an independent set. Given an independent set I , we call the graph $G' = G[V \setminus \Gamma_G[I]]$ the residual graph with respect to I . If clear from the context, we may simply call G' the residual graph. We say that a vertex $u \in V$ is *uncovered* with respect to I , if u is not adjacent to a vertex in I , i.e., $u \in V \setminus \Gamma_G[I]$. Again, if clear from the context, we simply say u is uncovered without specifying I explicitly.

Ghaffari gave the following result that we will reuse in this paper:

⁴ The authors of [1] kindly shared an extended version of their paper with me.

Theorem 2 (Ghaffari [8]). *Let G be a n -vertex graph with $\Delta(G) = \text{poly log}(n)$. Then there is a distributed algorithm that runs in the CONGESTED-CLIQUE model and computes a MIS on G in $O(\log \log \Delta)$ rounds.*

Routing. As a subroutine, our algorithm needs to solve the following simple routing task: Let $u \in V$ be an arbitrary vertex. Suppose that every other vertex $v \in V \setminus \{u\}$ holds $0 \leq n_v \leq n$ messages each of size $O(\log n)$ that it wants to deliver to u . We are guaranteed that $\sum_{v \in V} n_v \leq n$. Lenzen proved that in the CONGESTED-CLIQUE model there is a deterministic routing scheme that achieves this task in $O(1)$ rounds [18]. In the following, we will refer to this scheme as Lenzen’s routing scheme.

Concentration Bound for Dependent Variables. In the analysis of our algorithm, we require a Chernoff bound for dependent variables (see for example [6]):

Theorem 3 (Chernoff Bound for Dependent Variables, e.g. [6]). *Let X_1, X_2, \dots, X_n be 0/1 random variables for which there is a $p \in [0, 1]$ such that for all $k \in [n]$ and all $a_1, \dots, a_{k-1} \in \{0, 1\}$ the inequality*

$$\mathbb{P}[X_k = 1 \mid X_1 = a_1, X_2 = a_2, \dots, X_{k-1} = a_{k-1}] \leq p$$

holds. Let further $\mu \geq p \cdot n$. Then, for every $\delta > 0$:

$$\mathbb{P}\left[\sum_{i=1}^n X_i \geq (1 + \delta)\mu\right] \leq \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}}\right)^\mu.$$

Last, we say that an event occurs with high probability if the probability of the event not occurring is at most $\frac{1}{n}$.

3 Sequential Random Order Greedy Algorithm for MIS

The GREEDY algorithm for maximal independent set processes the vertices of the input graph in arbitrary order. It adds the current vertex under consideration to an initially empty independent set I if none of its neighbors are already in I .

This algorithm progressively thins out the input graph, and the rate at which the graph loses edges depends heavily on the order in which the vertices are considered. If the vertices are processed in uniform random order (Algorithm 1), then the number of edges in the residual graph decreases relatively quickly. A variant of the next lemma was proved in [1] in the context of correlation clustering in the streaming model:

Lemma 1. *Let t be an integer with $1 \leq t < n$. Let U_i be the set U at the beginning of iteration i of Algorithm 1. Then with probability at least $1 - n^{-9}$ the following holds:*

$$\Delta(G[U_t]) \leq 10 \ln(n) \frac{n}{t}.$$

Proof. Fix an arbitrary index $j \geq t$. We will prove that either vertex v_j is not in U_t , or it has at most $10 \ln(n) \frac{n}{t}$ neighbors in $G[U_t]$, with probability at least $1 - n^{-10}$. The result follows by a union bound over the error probabilities of all n vertices.

We consider the following process in which the random order of the vertices is determined. First, reveal v_j . Then, reveal vertices v_i just before iteration i of the algorithm. Let $N_i :=$

Input: $G = (V, E)$ is an n -vertex graph

1. Let v_1, v_2, \dots, v_n be a uniform random ordering of V
2. $I \leftarrow \{\}, U \leftarrow V$ (U is the set of uncovered elements)
3. **for** $i \leftarrow 1, 2, \dots, n$ **do**
 - if** $v_i \in U$ **then**
 - $I \leftarrow I \cup \{v_i\}$
 - $U \leftarrow U \setminus \Gamma_G[v_i]$
4. **return** I

Algorithm 1. Random order GREEDY algorithm for MIS.

$\Gamma_G(v_j) \cap U_i$ be the set of neighbors of v_j that are uncovered in the beginning of iteration i , and let $d_i = |N_i|$. For every $1 \leq i \leq t-1$, the following holds:

$$\mathbb{P}[v_i \in N_i \mid v_j, v_1, \dots, v_{i-1}] = \frac{d_i}{n-1-(i-1)} \geq \frac{d_i}{n},$$

since v_i can be one of the not yet revealed $n-1-(i-1)$ vertices. We now distinguish two cases. First, suppose that $d_{t-1} \leq 10 \ln(n) \frac{n}{t}$. Then the result follows immediately since, by construction, $d_t \leq d_{t-1}$ (the sequence $(d_i)_i$ is decreasing). Suppose next that $d_{t-1} > 10 \ln(n) \frac{n}{t}$. Then, we will prove that with high probability there is one iteration $i' \leq t-1$ in which a neighbor of v_j is considered by the algorithm, i.e., $v_{i'} \in N_{i'}$. This in turn implies that v_j is not in U_t . We have:

$$\begin{aligned} \mathbb{P}[\forall i < t : v_i \notin N_i \mid v_j] &\leq \prod_{i < t} \mathbb{P}[v_i \notin N_i \mid v_j, v_1, \dots, v_{i-1}] \leq \prod_{i < t} \left(1 - \frac{d_i}{n}\right) \\ &\leq \left(1 - \frac{d_{t-1}}{n}\right)^{t-1} \leq e^{-\frac{d_{t-1}(t-1)}{n}} \leq n^{-10}. \end{aligned}$$

□

4 MIS Algorithm in the Congest Clique Model

4.1 Algorithm

Our MIS algorithm, depicted in Algorithm 2, consists of three parts:

First, all vertices agree on a uniform random order as follows. The vertex with the smallest ID chooses a uniform random order locally and informs all other vertices about their positions within the order. Then, all vertices broadcast their positions to all other vertices. As a result, all vertices know the entire order. Let v_1, v_2, \dots, v_n be this order.

Next, we simulate GREEDY until the maximum degree of the residual graph is at most $\log^4 n$ (this bound is chosen only for convenience; any poly-logarithmic number in n is equally suitable). To this end, in each iteration of the while-loop, we first determine a number k as a function of the maximum degree $\Delta(G')$ of the current residual graph G' so that the subgraph of G' induced by the yet uncovered vertices of $\{v_1, \dots, v_k\}$ has at most n edges w.h.p. (see Lemma 3). Using Lenzen's routing protocol, these edges are collected at vertex v_1 , which continues the simulation of GREEDY up to iteration k . It then informs the chosen vertices about their selection, who in turn inform their neighbors about their selection. Vertices then compute the new residual graph and its maximum degree and proceed with the next iteration of the while-loop. We prove

Input: $G = (V, E)$ is an n -vertex graph with maximum degree $\Delta := \Delta(G)$
Set parameter $C = 5$

1. Nodes agree on random order.

All vertices exchange their IDs in one round. Let $u \in V$ be the vertex with the smallest ID. Vertex u chooses a uniform random order of V and informs every vertex $v \in V \setminus \{u\}$ about its position r_v within the order. Then, every vertex $v \in V$ broadcasts r_v to all other vertices. As a result, all vertices know the order. Let v_1, v_2, \dots, v_n be the resulting order.

2. Simulate sequential Greedy.

Every vertex v_i sets $u_i \leftarrow true$ indicating that v_i is uncovered. Let $G' := G$.

Every vertex v_i broadcasts $\deg_{G'}(v_i)$ to all other vertices so that every vertex knows $\Delta(G')$.

while $\Delta(G') > \log^4 n$ **do**

(a) Let $k \leftarrow \frac{n}{\sqrt{\Delta(G')C}}$

(b) Every vertex v_i with $u_i = true$ and $i \leq k$ sends all its incident edges $v_i v_j$ with $u_j = true$ and $j < i$ to v_1 using Lenzen's routing protocol in $O(1)$ rounds.

(c) Vertex v_1 knows the subgraph H of uncovered vertices v_j with $j \leq k$, i.e.,

$$H := G'[\{v_j : j \leq k \text{ and } u_j = true\}] .$$

It continues the simulation of GREEDY up to iteration k using H . Let I' be the vertices selected into the independent set.

(d) Vertex v_1 informs nodes I' about their selection in one round. Nodes I' inform their neighbors about their selection in one round.

(e) Every node $v_i \in I_G[I']$ sets $u_i \leftarrow false$.

(f) Let $G' := G[\{v_i \in V : u_i = true\}]$. Every vertex v_i broadcasts u_i to all other vertices. Then every vertex v_i computes $\deg_{G'}(v_i)$ locally and broadcasts $\deg_{G'}(v_i)$ to all other vertices. As a result, every vertex knows $\Delta(G')$.

end while

3. Run Ghaffari's algorithm.

Run Ghaffari's MIS algorithm on G' in $O(\log \log \Delta)$ rounds.

Algorithm 2. $O(\log \log \Delta)$ rounds MIS algorithm in the CONGESTED-CLIQUE model.

in Lemma 2 that only $O(\log \log \Delta)$ iterations of the while-loop are necessary until $\Delta(G')$ drops below $\log^4 n$.

Last, we run Ghaffari's algorithm on G' which completes the maximal independent set computation.

4.2 Analysis

Let G'_i denote the graph G' at the beginning of iteration i of the while-loop. Notice that $G'_1 = G$. Let $\Delta_i := \Delta(G'_i)$ and let $k_i = \frac{n}{\sqrt{\Delta_i C}}$ be the value of k in iteration i . Observe that the while-loop is only executed if $\Delta_i > \log^4 n$ and hence

$$k_i \geq \frac{n}{\log^2 n C} \tag{1}$$

holds for every iteration i of the while-loop. Further let H_i be the graph H in iteration i of the while-loop.

To establish the runtime of our algorithm, we need to bound the number of iterations of the while-loop. To this end, in the next lemma we bound Δ_i for every $1 \leq i \leq n$ and conclude that $\Delta_j \leq \log^4 n$, for some $j \in O(\log \log \Delta)$.

Lemma 2. *With probability at least $1 - n^{-8}$, for every $i \leq n$, the maximum degree in G'_i is bounded as follows:*

$$\Delta_i \leq \Delta^{\frac{1}{2^{i-1}}} \cdot 100C \ln^2 n .$$

Proof. We prove the statement by induction. Observe that $\Delta_1 = \Delta$ and the statement is thus trivially true for $i = 1$. Suppose that the statement holds up to some index $i - 1$. Recall that G'_i is the residual graph obtained by running GREEDY on vertices $v_1, \dots, v_{k_{i-1}}$. Hence, by applying Lemma 1, the following holds with probability $1 - n^{-9}$:

$$\Delta_i \leq 10 \ln(n) \frac{n}{k_{i-1}} = \frac{10 \ln(n) n}{\frac{n}{\sqrt{\Delta_{i-1} C}}} = \sqrt{\Delta_{i-1}} \cdot 10C \ln n .$$

Resolving the recursion, we obtain

$$\Delta_i = \Delta^{\frac{1}{2^{i-1}}} \cdot \prod_{j=0}^{i-2} (10C \ln n)^{\frac{1}{2^j}} = \Delta^{\frac{1}{2^{i-1}}} \cdot (10C \ln n)^{\sum_{j=0}^{i-2} \frac{1}{2^j}} \leq \Delta^{\frac{1}{2^{i-1}}} \cdot 100C^2 \ln^2 n .$$

Observe that we invoked n times Lemma 1. Thus, by the union bound, the result holds with probability $1 - n^{-8}$. \square

Corollary 1. $\Delta_i = O(\log^2 n)$ for some $i \in O(\log \log \Delta)$.

To establish correctness of the algorithm, we need to ensure that we can apply Lenzen's routing protocol to collect the edges of H_i at vertex v_1 . For this to be feasible, we need to prove that, for every i , H_i contains at most n edges with high probability.

Lemma 3. *With probability at least $1 - n^{-9}$, graph H_i has at most n edges.*

Proof. Let U_i be the vertex set of G'_i , i.e., the set of uncovered vertices at the beginning of iteration i . We will prove now that, with probability at least $1 - n^{-10}$, for every $v_j \in U_i$, the following holds

$$d(v_j) := |E_{G'_i}(v_j) \cap \{v_{k_{i-1}+1}, \dots, v_k\}| \leq \frac{n}{k_i} . \quad (2)$$

Since the vertex set of H_i is a subset of at most $k_i - k_{i-1} \leq k_i$ vertices of U_i , the result follows by applying the union bound on the error probabilities for every vertex of G'_i .

To prove Inequality 2, observe that graph G'_i is solely determined by vertices $v_1, v_2, \dots, v_{k_{i-1}}$, and the execution of the algorithm so far was not affected by the outcome of the random variables $v_{k_{i-1}+1}, \dots, v_n$. Thus, by the principle of deferred decision, for every $k_{i-1} + 1 \leq l \leq k_i$, vertex v_l can be seen as a uniform random vertex chosen from $V \setminus \{v_1, \dots, v_{l-1}\}$.

For $1 \leq l \leq k_i - k_{i-1}$, let X_l be the indicator variable of the event " $v_{k_{i-1}+l} \in E_{G'_i}(v_j)$ ". Observe that $d(v_j) = \sum_l X_l$ and

$$\mathbb{E}[d(v_j)] = \deg_{G'_i}(v_j) \cdot \frac{k_i - k_{i-1}}{n - k_{i-1}} \leq \deg_{G'_i}(v_j) \cdot \frac{k_i}{n} . \quad (3)$$

Furthermore, observe that for every $1 \leq l \leq k_i - k_{i-1}$, and all $a_1, \dots, a_{l-1} \in \{0, 1\}$, the inequality

$$\mathbb{P}[X_l = 1 \mid X_1 = a_1, X_2 = a_2, \dots, X_{l-1} = a_{l-1}] \leq \frac{\deg_{G'_i}(v_j)}{n - k_i} \leq \frac{2 \cdot \deg_{G'_i}(v_j)}{n}$$

holds (using the bound $k_i \leq n/2$, which follows from Inequality 1), since in the worst case, we have $a_1 = a_2 = \dots = a_{l-1} = 0$, which implies that there are $\deg_{G'_i}(v_j)$ choices left out of at least $n - k_i$ possibilities such that $X_l = 1$. We can thus use the Chernoff bound for dependent variables as stated in Theorem 3 in order to bound the probability that $d(v_j)$ deviates from its expectation.

We distinguish two cases. First, suppose that $\mathbb{E}[d(v_j)] \geq 4 \log n$. Then by Theorem 3 (setting $\mu = 2\mathbb{E}[d(v_j)]$ and $\delta = 8$),

$$\mathbb{P}[d(v_j) \geq 18 \cdot \mathbb{E}[d(v_j)]] \leq \exp\left(\frac{e^8}{(1+8)^{1+8}}\right)^{8 \log n} \leq n^{-10}.$$

Thus, using Inequality 3, with high probability,

$$d(v_j) \leq 18 \cdot \mathbb{E}[d(v_j)] \leq 18 \cdot \deg_{G'_i}(v_j) \frac{k_i}{n} \leq 18 \cdot \frac{\Delta_i}{\sqrt{\Delta_i} C} \leq 18 \cdot \frac{n}{k_i C^2} \leq \frac{n}{k_i},$$

since $C \geq 5$. Suppose now that $\mathbb{E}[d(v_j)] < 4 \log n$. Then, by Theorem 3 (setting $\mu = 8 \log n$ and $\delta = 8$),

$$\mathbb{P}[d(v_j) \geq 72 \log n] \leq n^{-10},$$

by the same calculation as above. Since $k_i \leq \frac{n}{\log^2 n \cdot C}$ (Inequality 1), we have $d(v_j) \leq \frac{n}{k_i}$, which completes the proof. \square

Theorem 1 (restated) *Algorithm 2 operates in $O(\log \log \Delta)$ rounds in the CONGESTED-CLIQUE model and outputs a maximal independent set with high probability.*

Proof. Concerning the runtime, Step 1 of the algorithm requires $O(1)$ communication rounds. Observe that every iteration of the while-loop requires $O(1)$ rounds. The while-loop terminates in $O(\log \log \Delta)$ rounds with high probability, by Corollary 1. Since Ghaffari's algorithm requires $O(\log \log \Delta') = O(\log \log \Delta)$ rounds, where Δ' is the maximum degree in the residual as computed in the last iteration of the while-loop (or in case $\Delta < \log^4 n$ then $\Delta' = \Delta$), the overall runtime is bounded by $O(\log \log \Delta)$.

Concerning the correctness of the algorithm, the only non-trivial step is the collection of graph H_i at vertex v_1 . This is achieved using Lenzen's routing protocol, which can be used since we proved in Lemma 3 that graph H_i has at most n vertices with high probability. \square

5 Conclusion

In this paper, we gave a $O(\log \log \Delta)$ rounds MIS algorithm that runs in the CONGESTED-CLIQUE model. We simulated the sequential random order GREEDY algorithm, exploiting the residual sparsity property of GREEDY.

It is conceivable that the round complexity can be reduced further - there are no lower bounds known for MIS in the CONGESTED-CLIQUE model. Results on other problems, such as the minimum weight spanning tree problem where the $O(\log \log n)$ rounds algorithm of Lotker et al. [21] has subsequently been improved to $O(\log \log \log n)$ rounds [10], $O(\log^* n)$ rounds [9], and finally to $O(1)$ rounds [13], give hope that similar improvements may be possible for MIS as well. Can we simulate other centralized Greedy algorithms in few rounds in the CONGESTED-CLIQUE model?

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