C^1 analysis of 2D subdivision schemes refining point-normal pairs with the circle average

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Abstract

This article continues the investigation started in [9] on subdivision schemes refining 2D point-normal pairs, obtained by modifying linear subdivision schemes using the circle average. While in [9] the convergence of the Modified Lane-Riesenfeld algorithm and the Modified 4-Point schemes is proved, here we show that the curves generated by these two schemes are C^1.

Keywords: subdivision of 2D point-normal pairs, circle average, 2D curve design, modified Lane-Riesenfeld algorithm, modified 4-point scheme, smoothness analysis by proximity

1 Introduction

Subdivision schemes generate smooth curves/surfaces from discrete data by repeated refinements. Schemes based on linear averaging are widely used in applications, such as Computer Graphics and Computer Aided Geometric Design. The typical input to these schemes consists of a mesh of vertices. For a comprehensive overview of linear subdivision schemes see e.g. [6]. Another family of linear schemes, the Hermite subdivision schemes, refine function and its derivative values [10], [4]. Later, linear schemes were adapted to refine other types of geometric objects such as sets, manifold-valued data, and nets of functions (see e.g. [5], [11], [12], [3]). In recent years this list has been extended further to combined types of input data, such as point-normal pairs (e.g. [2], [1], [8]) and point-tangent pairs (e.g. [13]). The analysis of these new algorithms requires new tools and techniques (e.g. [12], [7]).

In this paper we consider subdivision schemes refining 2D point-normal pairs (PNPs), which are obtained from converging linear subdivision schemes, expressed in terms of repeated binary linear averages of points. We replace these averages by circle averages of PNPs. The so obtained schemes are termed "Modified schemes".

Two schemes are of special interest to us: the modified Lane-Riesenfeld (MLR) algorithm and the modified 4-Point (M4Pt) scheme. In [9] we proved their convergence, and here we prove that their limits are C^1. The analysis is

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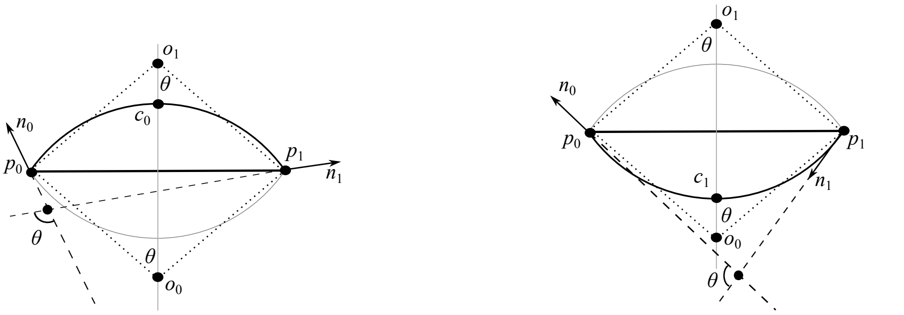


(a) n_0 and n_1 are in the same half-plane: $p_{\frac{1}{2}} = c_0$.

(b) n_0 and n_1 are in different half-planes: $p_{\frac{1}{2}} = c_1$.

Figure 1: Construction of $P_0 \circledcirc_{\frac{1}{2}} P_1$.

based on the Proximity tools from [12]. We prove several lemmas that facilitate the application of the Proximity tool to the MLR and the M4Pt.

The paper has the following structure.

In section 2 we recall the definition of the circle average, mention briefly some of its properties proven in [9], and present the algorithms MLR and M4Pt. The C^1 analysis is presented in section 3. In section 4, we propose a plausible generalization of this analysis.

2 The circle average in 2D and the modified subdivision algorithms based on it

In this section we recall the definition of the *circle average* - a weighted binary average of two 2D point-normal pairs, and present the algorithms for MLR and M4Pt.

2.1 Construction of the circle average

Given two pairs in 2D, each consisting of a point and a normal unit vector, $P_0 = (p_0, n_0)$ and $P_1 = (p_1, n_1)$, and a real weight $\omega \in [0, 1]$, the circle average produces a new pair $P_\omega = P_0 \circledcirc_\omega P_1 = (p_\omega, n_\omega)$.

To present the operation $P_0 \circledcirc_\omega P_1$ we introduce some notation. The angle $\theta(u, v)$ denotes the angle between the vectors u and v. In the special case of $u = n_0$ and $v = n_1$, the symbol θ substitutes $\theta(n_0, n_1)$. Observe that $0 \leq \theta \leq \pi$. The length of the segment $[p_0, p_1]$ is denoted by $|p_0 p_1|$, and $\overrightarrow{p_0 p_1}$ denotes the vector $\overrightarrow{p_1 - p_0}$. For two unit vectors $u = (\cos\alpha, \sin\alpha), v = (\cos\beta, \sin\beta)$, regarded as points on the unit circle, we denote by $GA(u, v; \omega)$ their weighted geodesic average given by

$$GA(u, v; \omega) = (\cos\gamma, \sin\gamma),\ \gamma = (1 - \omega)\alpha + \omega\beta. \tag{1}$$

The construction of $P_\omega = \{p_\omega, n_\omega\} = P_0 \circledcirc_\omega P_1$ is done in several steps.

1. Construct the perpendicular $[p_0, p_1]^\perp$ to the segment $[p_0, p_1]$ at its midpoint. Construct two circles with centers o_0 and o_1 on $[p_0, p_1]^\perp$, passing

Algorithm 1 MLR

Input: $m \in \mathbb{N}_0,\ P_i = (p_i, n_i),\ i \in \mathbb{Z}$.

for $i \in \mathbb{Z}$ **do**
  $P_i^0 \leftarrow P_i$
end for
for j=1,2,... **do**
  $\left.\begin{array}{l}\textbf{for } i \in \mathbb{Z} \textbf{ do}\\ \quad P_{2i}^{j,0} \leftarrow P_i^{j-1}\\ \quad P_{2i+1}^{j,0} \leftarrow P_i^{j-1} \circledcirc_{\frac{1}{2}} P_{i+1}^{j-1}\\ \textbf{end for}\text{(i)}\end{array}\right\}$ elementary refinement step
  for $k = 1, \ldots, m-1$ **do**
    $\left.\begin{array}{l}\textbf{for } i \in \mathbb{Z} \textbf{ do}\\ \quad P_i^{j,k} \leftarrow P_i^{j,k-1} \circledcirc_{\frac{1}{2}} P_{i+1}^{j,k-1}\\ \textbf{end for}\text{(i)}\end{array}\right\}$ smoothing step
  end for(k)
  $\left.\begin{array}{l}\textbf{for } i \in \mathbb{Z} \textbf{ do}\\ \quad P_i^{j} \leftarrow P_i^{j,m-1}\\ \textbf{end for}\text{(i)}\end{array}\right\}$ result of current iteration
end for(j)

through p_0 and p_1, so that the central angles $\sphericalangle p_0 o_i p_1, i = 0, 1$ equal θ. Note that the two circles are symmetric relative to the segment $[p_0, p_1]$, with the same radius $\frac{|p_0 p_1|}{2 sin \frac{\theta}{2}}$.

2. Use the **Selection Criterion** (See [9]) to choose one of the circles. Take the short arc connecting p_0 and p_1 on the selected circle. (We denote the center of the selected circle by o^*.)

3. Compute p_ω on the short arc such that $\sphericalangle p_0 o^* p_\omega = \omega\theta$.

4. Take the normal n_ω as $GA(n_0, n_1; \omega)$.

Two examples of the construction are given in Figure 1. The candidate arcs in both cases are the same, since in both cases θ is the same.

Note that $P_0 \circledcirc_0 P_1 = P_0$ and $P_0 \circledcirc_1 P_1 = P_1$. In [9] we show that $P_0 \circledcirc_\omega P_1$ is indeed a weighted average. Namely, that

$$\forall t, s, k \in [0,1], (P_0 \circledcirc_t P_1) \circledcirc_k (P_0 \circledcirc_s P_1) = P_0 \circledcirc_{\omega^*} P_1,\ \omega^* = ks + (1-k)t \quad (2)$$

We call this property the ***consistency*** property.

2.2 The modified algorithms

Here we provide the pseudo code of two modified algorithms with the circle average. The first is the modification of the Lane-Riesenfeld algorithm. Note that in this case, the linear algorithm for points is already given in the form of repeated binary averages. The modified scheme is presented in Algorithm 1.

To modify the linear 4-point scheme we first write the insertion rule in terms of repeated binary averages in a symmetric way. The modified scheme is presented in Algorithm 2.

Algorithm 2 M4Pt

Input: $P_i = (p_i, n_i),\ i \in \mathbb{Z}$.

for $i \in \mathbb{Z}$ **do**
  $P_i^0 \leftarrow P_i$
end for
for j=1,2,... **do**
  for $i \in \mathbb{Z}$ **do**
    $P_{2i}^j \leftarrow P_i^{j-1}$
    $S_L \leftarrow P_i^{j-1} \circledcirc_{-\frac{1}{8}} P_{i-1}^{j-1}$
    $S_R \leftarrow P_{i+1}^{j-1} \circledcirc_{-\frac{1}{8}} P_{i+2}^{j-1}$
    $P_{2i+1}^j \leftarrow S_L \circledcirc_{\frac{1}{2}} S_R$
  end for
end for

3 Smoothness analysis

In this section we study the smoothness of the limit curves generated by the MLR and the M4Pt. To show the C^1 smoothness of the curves generated by these two schemes we use the Proximity tool of [12].

3.1 Auxiliary results

First, we state the proximity result relevant to our analysis, and then we derive several geometric results which facilitate the application of the proximity tool.

Result A (Smoothness from proximity [12])

Let S be a converging linear scheme operating on control points with C^1 limits and satisfying the technical condition (41) in [12]. Let T be a converging non-linear scheme operating on control points.

If for all polygonal lines $\mathcal{P} = \{p_i \in \mathbb{R}^d, i \in \mathbb{Z}\}$, satisfying

$$d(\mathcal{P}) = \sup_i \{|p_{i+1}p_i|\} \leq \delta, \tag{3}$$

we have

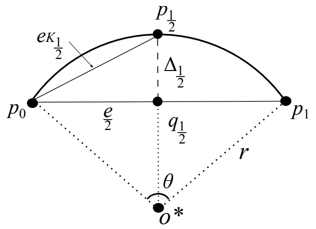
$$d(T\mathcal{P} - S\mathcal{P}) \leq C(d(\mathcal{P}))^2 \tag{4}$$

with C a constant, then the limit curves generated by T are C^1.

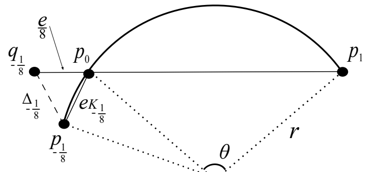
We introduce some additional notation. For given P_0, P_1 two PNPs, $P_i = (p_i, n_i), i = 0, 1$, we denote by q_ω the linear average $q_\omega = (1-\omega)p_0 + \omega p_1$, by p_ω the point of $P_0 \circledcirc_\omega P_1$, and by e the length of the segment $[p_0, p_1]$.

We study the distance $\Delta_\omega = |p_\omega q_\omega|$ in the cases $\omega = \frac{1}{2}$ and $\omega = -\frac{1}{8}$, by applying the cosine theorem in the triangle $p_\omega p_0 q_\omega$ (see Figure 2). Since for any $\omega \in [-\frac{1}{8}, \frac{1}{2}]$

$$|p_0 p_\omega| = e\frac{\sin\left(\frac{\theta|\omega|}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}, \text{ and } \sphericalangle p_\omega p_0 q_\omega = \frac{\theta\,(1-\omega)}{2}, \tag{5}$$



(a) $\omega = \frac{1}{2}$



(b) $\omega = -\frac{1}{8}$

Figure 2: Studying Δ_ω.

we get

$$\Delta_\omega^2 = (e|\omega|)^2 + \left(e\frac{\sin\left(\frac{\theta|\omega|}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\right)^2 - 2e^2|\omega|\frac{\sin\left(\frac{\theta|\omega|}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}\cos\left(\frac{\theta\,(1-\omega)}{2}\right). \tag{6}$$

Denoting $\kappa_\omega = \dfrac{\sin\left(\frac{\theta|\omega|}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$ and rewriting (6), we get

$$\Delta_\omega^2 = (e|\omega|)^2 + (e\kappa_\omega)^2 - 2e^2|\omega|\kappa_\omega\cos\left(\frac{\theta\,(1-\omega)}{2}\right). \tag{7}$$

It is easy to see from the Taylor expansion of $\cos x$ that for $x \in [0, \frac{\pi}{2}]$

$$\cos x \geq 1 - \frac{x^2}{2}. \tag{8}$$

Thus

$$\Delta_\omega^2 \leq (e|\omega|)^2 + (e\kappa_\omega)^2 - 2e^2|\omega|\kappa_\omega\Big[1 - \frac{1}{2}\Big(\frac{\theta(1-\omega)}{2}\Big)^2\Big], \tag{9}$$

and after simplifications

$$\Delta_\omega^2 \leq (e|\omega| - e\kappa_\omega)^2 + e^2|\omega|\kappa_\omega\Big(\frac{\theta(1-\omega)}{2}\Big)^2. \tag{10}$$

In this paper we derive κ_ω for $\omega = 2^{-n}$ and apply it to $|\omega| = \frac{1}{2}, \frac{1}{8}$ in the MLR and M4Pt. Using the trigonometric identity $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ for the denominator of κ_ω repeatedly, we get

$$\kappa_{2^{-n}} = \frac{\sin\frac{\theta}{2^{n+1}}}{\sin\frac{\theta}{2}} = \frac{1}{2^n\displaystyle\prod_{i=2}^{n+1}\cos\frac{\theta}{2^i}}. \tag{11}$$

Using (8) again, and the inequality $\frac{1}{1-x} \leq 1 + 2x$, for $0 \leq x \leq \frac{1}{2}$, we get

$$\kappa_{2^{-n}} \leq \frac{1}{2^n\displaystyle\prod_{i=2}^{n+1}\Big(1 - \frac{1}{2}\Big(\frac{\theta}{2^i}\Big)^2\Big)} \leq \frac{1}{2^n}\prod_{i=2}^{n+1}\Big(1 + \Big(\frac{\theta}{2^i}\Big)^2\Big). \tag{12}$$

Note that by (11), $2^{-n} < \kappa_{2^{-n}}$ and therefore we can replace κ_ω in (10) by its bound (12). Inserting (12) into (10), and using $\theta \leq \pi$, we arrive at

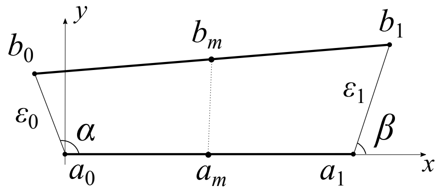


Figure 3: The setup of Lemma 3.2.

Lemma 3.1.

$$\Delta_{2^{-n}} \leq \chi_{2^{-n}} \theta e, \tag{13}$$

where $\chi_{2^{-n}}$ *is a constant depending on* n*.*

The next lemma shows the stability of the midpoint of two points.

Lemma 3.2. *For two segments* $[a_0, a_1]$ *and* $[b_0, b_1]$ *in 2D, with midpoints* a_m *and* b_m *respectively,*

$$|a_m b_m| \leq 2 \max\{\epsilon_0, \epsilon_1\},$$

where $\epsilon_0 = |a_0 b_0|$ *and* $\epsilon_1 = |a_1 b_1|$*.*

Proof. We introduce a local coordinate system. Let $a_0 = (0,0)$ and let the $x-$axis pass through a_1. See Figure 3 for an example. We denote the angle between $[a_0, b_0]$ ($[a_1 b_1]$) and the $x-$axis by α (β), where $-\pi \leq \alpha, \beta \leq \pi$. Then the coordinates of the points are

$$a_0 = (0,0), \quad a_1 = (|a_0 a_1|, 0),$$

$$b_0 = (\epsilon_0 \cos\alpha, \epsilon_0 \sin\alpha), \quad b_1 = (|a_0 a_1| + \epsilon_1 \cos\beta, \epsilon_1 \sin\beta),$$

$$a_m = \Big(\frac{|a_0 a_1|}{2}, 0\Big), \quad b_m = \Big(\frac{|a_0 a_1|}{2} + \frac{\epsilon_1 \cos\beta}{2} + \frac{\epsilon_0 \cos\alpha}{2}, \frac{\epsilon_1 \sin\beta}{2} + \frac{\epsilon_0 \sin\alpha}{2}\Big).$$

Thus

$$\begin{aligned} |a_m b_m| &= \sqrt{\Big(\frac{|a_0 a_1|}{2} - \frac{|a_0 a_1|}{2} - \frac{\epsilon_1 \cos\beta}{2} - \frac{\epsilon_0 \cos\alpha}{2}\Big)^2 + \Big(\frac{\epsilon_1 \sin\beta}{2} + \frac{\epsilon_0 \sin\alpha}{2}\Big)^2} \\ &\leq \frac{1}{2} \max\{\epsilon_0, \epsilon_1\} \sqrt{\Big(\cos\beta + \cos\alpha\Big)^2 + \Big(\sin\beta + \sin\alpha\Big)^2} \\ &\leq 2 \max\{\epsilon_0, \epsilon_1\} \end{aligned}$$

□

Both the Lane-Riesenfeld algorithm and the 4-point scheme satisfy the conditions required by Result A. The application of the proximity tool to the modified schemes which refine sequences of PNPs is possible due to the convergence of the normals, as is demonstrated in the next section.

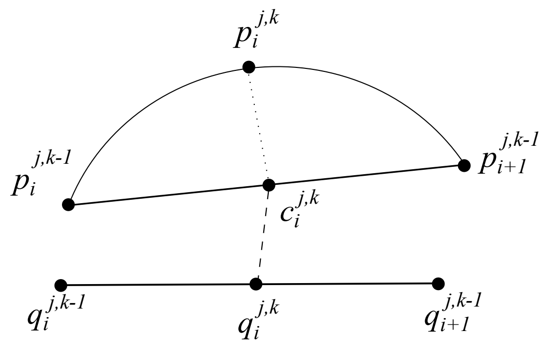


Figure 4: The setup of Lemma 3.3.

3.2 Smoothness of the MLR algorithm

In the notation of Algorithm 1, we define

$$e_i^{j,k} = |p_i^{j,k} p_{i+1}^{j,k}|, \quad e^{j,k} = \max_{i\in\mathbb{Z}} e_i^{j,k}, \tag{14}$$

$$\theta_i^{j,k} = \theta(n_i^{j,k}, n_{i+1}^{j,k}), \quad \theta^{j,k} = \max_{i\in\mathbb{Z}} \theta_i^{j,k}, \tag{15}$$

where $P_i^{j,k} = (p_i^{j,k}, n_i^{j,k})$, and

$$e^j = e^{j,m-1}, \ \theta^j = \theta^{j,m-1}, \mathcal{P}^j = \{p_i^{j,m-1} : i \in \mathbb{Z}\} \tag{16}$$

We know from the convergence of the MLR [9] that $\lim\limits_{j\to\infty} e_i^{j,k} = 0$ and $\lim\limits_{j\to\infty} \theta_i^{j,k} = 0$.
Denote by $q_i^{j,k}, i \in \mathbb{Z}$, the points obtained by the k^{th} step of the j^{th} iteration of the linear Lane-Riesenfeld algorithm operating on $\mathcal{P}^{j-1}$.
Let $\Delta_i^{j,k} = |q_i^{j,k} p_i^{j,k}|$ and let $\Delta^{j,k} = \sup_i \Delta_i^{j,k}$.

Lemma 3.3. *In the above notation, for $k \geq 1$, we have*

$$\Delta_i^{j,k} \leq 2\Delta^{j,k-1} + \chi_{\frac{1}{2}} e_i^{j,k-1} \theta_i^{j,k-1} \tag{17}$$

Proof. We consider an auxiliary point $c_i^{j,k}$, which is the midpoint of the segment $[p_i^{j,k-1}, p_{i+1}^{j,k-1}]$. See Figure 4 for an example. Using the triangle inequality, we get

$$\Delta_i^{j,k} \leq |q_i^{j,k} c_i^{j,k}| + |c_i^{j,k} p_i^{j,k}|. \tag{18}$$

By Lemma 3.2 a bound on the first term in (18) is

$$|q_i^{j,k} c_i^{j,k}| \leq 2\Delta^{j,k-1}. \tag{19}$$

To bound the second term in (18), we apply Lemma 3.1

$$|c_i^{j,k} p_i^{j,k}| \leq \chi_{\frac{1}{2}} e_i^{j,k-1} \theta_i^{j,k-1}. \tag{20}$$

Combining (18), (19) and (20), we arrive at (17). $\square$

Theorem 3.4. *The limit curves of the MLR algorithm are C^1, for $m \geq 2$.*

Proof. We consider one MLR iteration. It consists of the elementary refinement step and $m-1$ smoothing steps. In the elementary refinement step, the distance between the new vertices inserted by the linear average and by the circle average is bounded by the expression of Lemma 3.1. Thus

$$\Delta_{2i+1}^{j,0} \leq \chi_{\frac{1}{2}} e_i^{j,0} \theta_i^{j,0}, \ \Delta_{2i}^{j,0} = 0, \ i \in \mathbb{Z}. \tag{21}$$

For a smoothing step, we apply Lemma 3.3 and obtain for $1 \leq k \leq m-1$

$$\Delta_i^{j,k} \leq 2\Delta^{j,k-1} + \chi_{\frac{1}{2}} e^{j,k-1} \theta^{j,k-1}. \tag{22}$$

This leads to

$$\Delta^{j,k} \leq 2^k \Delta^{j,0} + \chi_{\frac{1}{2}} \sum_{\ell=0}^{k-1} 2^{k-1-\ell} e^{j,\ell} \theta^{j,\ell}. \tag{23}$$

From [9], Lemma 3.1 and its proof we get that for j large enough

$$e^{j,k} \leq e^{j,0} = e^{j-1} \Rightarrow e^j \leq e^{j-1}, \tag{24}$$

$$\theta^{j,k} \leq \theta^{j,0} = \theta^{j-1} \Rightarrow \theta^j \leq \theta^{j-1}. \tag{25}$$

Since $e^{j,k} \leq e^{j,k-1}$ and $\theta^{j,k} \leq \theta^{j,k-1}$, the bound on $\Delta^{j,m-1}$ becomes

$$\Delta^{j,m-1} \leq 2^{m-1} \Delta^{j,0} + \chi_{\frac{1}{2}} (2^{m-1} - 1) e^{j-1} \theta^{j-1}. \tag{26}$$

We showed in [9] that as $j \to \infty$,

$$\frac{\theta^j}{\theta^{j-1}} \to \frac{1}{2}, \quad \frac{e^j}{e^{j-1}} \to \frac{1}{2} \Big(\frac{1}{\cos \frac{\theta^{j-1}}{4}} \Big)^m > \frac{1}{2}. \tag{27}$$

Thus θ^j approaches zero faster than e^j. and for j large enough $\theta^j < e^j$. Substituting θ^j with e^j in (26) and using (21) we get, in view of (24) - (25), for j large enough

$$\Delta^j = \max_i |q_i^j p_i^j| \leq \Big(\chi_{\frac{1}{2}} 2^{m-1} + \chi_{\frac{1}{2}} (2^{m-1} - 1) \Big) \Big(e^{j-1} \Big)^2. \tag{28}$$

It follows from (27) that for j large enough $e^{j-1} \leq 2e^j$, and we arrive at

$$\Delta^j \leq 2^{m+2} \chi_{\frac{1}{2}} \Big(e^j \Big)^2. \tag{29}$$

This constitutes the proximity condition (4) between the operation of the linear Lane-Riesenfeld algorithm and the MLR, on control points, generated by the MLR at level j for j large enough. By the convergence of the MLR for j large enough (3) holds. Since the linear LR is C^1 for $m \geq 2$, we obtain, in view of Result A, that the limit curves of the MLR algorithm are C^1 for $m \geq 2$. $\square$

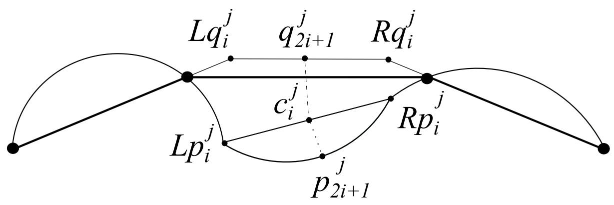


Figure 5: The setup of Theorem 3.5.

3.3 Smoothness of the M4Pt scheme

Theorem 3.5. *The limit curves of the M4Pt algorithm are* C^1

Proof. As in Algorithm 2, one iteration of the linear 4-point scheme, written in terms of repeated binary averages, consists of two extrapolating steps and one averaging step. The extrapolating steps produce points Lq_i^j (left) and Rq_i^j (right) by the linear 4-point scheme, and Lp_i^j (left) and Rp_i^j (right) by the M4Pt. The midpoint of $[Lq_i^j, Rq_i^j]$ is q_{2i+1}^j. We denote by c_i^j the midpoint of $[Lp_i^j, Rp_i^j]$, and by $L\Delta_i^j = |Lp_i^j Lq_i^j|$, $R\Delta_i^j = |Rp_i^j Rq_i^j|$. See Figure 5 for an example. By the triangle inequality,

$$|q_{2i+1}^j p_{2i+1}^j| \leq |q_{2i+1}^j c_i^j| + |c_i^j p_{2i+1}^j|, \tag{30}$$

and by Lemma 3.2

$$|q_{2i+1}^j c_i^j| \leq 2 \max\{L\Delta_i^j, R\Delta_i^j\}. \tag{31}$$

Using Lemma 3.1, we get

$$|q_{2i+1}^j c_i^j| \leq 2\chi_{-\frac{1}{8}} e^j \theta^j, \tag{32}$$

where

$$\theta^j = \max_i \theta(n_i^j, n_{i+1}^j),\ e^j = \max_i |p_i^j, p_{i+1}^j| \tag{33}$$

Let n_L^j and n_R^j be the normals associated with Lp_i^j and Rp_i^j respectively. We showed in [9] that for every j $\theta(n_L^j, n_R^j) \leq \frac{5}{4}\theta^j$ and $|Lp_i^j Rp_i^j| \leq \frac{7}{8}e^j$. We use these inequalities and Lemma 3.1 to bound the second term in the right-hand side of (30),

$$|c_i^j p_{2i+1}^j| \leq \chi_{\frac{1}{2}} \frac{7}{8} e^j \frac{5}{4} \theta^j = \frac{35}{32} \chi_{\frac{1}{2}} e^j \theta^j. \tag{34}$$

Inserting(32) and (34) into (30) and considering j large enough, we obtain

$$\max_i |q_i^j p_i^j| \leq \big(2\chi_{-\frac{1}{8}} + \frac{35}{32}\chi_{\frac{1}{2}}\big) e^j \theta^j. \tag{35}$$

We proved in [9] that in the M4Pt as $j \to \infty$, $\frac{\theta^{j+1}}{\theta^j} \to \frac{5}{8}$, and that $\frac{e^{j+1}}{e^j} \to \frac{6}{8}$. I.e. that θ^j approaches zero faster than e^j. We apply this fact in (35) and get, for j large enough

$$\max_i |q_i^j p_i^j| \leq \big(2\chi_{-\frac{1}{8}} + \frac{35}{32}\chi_{\frac{1}{2}}\big) \Big(e^j\Big)^2. \tag{36}$$

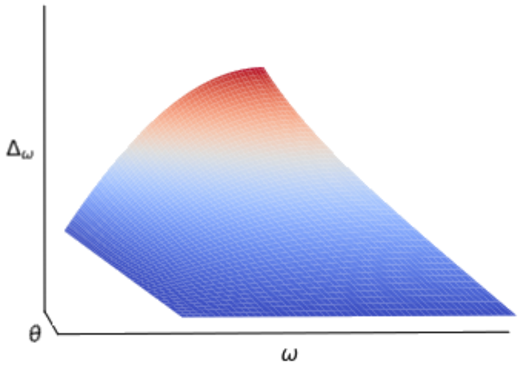


Figure 6: Graph of Δ_ω for $e = 1$, $\omega \in [0, \frac{1}{2}]$, $\theta \in [0, \pi)$

So, for j large enough, the M4Pt scheme is in proximity with the linear 4-point scheme. Since the M4Pt scheme is converging, $e^j < \delta$ for j large enough, namely (3) holds for the control points generated by the M4Pt. Due to the fact that the linear 4-point scheme has C^1 limit, Result A implies that the M4Pt has that property too. □

4 Possible extension of Theorems 3.4, 3.5

In this section we consider general linear subdivision scheme and their modifications. A general linear scheme is given by the refinement rule [6]

$$f_i^{j+1} = \sum_{k\in\mathbb{Z}} a_{i-2k} f_k^j, \quad j \geq 0. \tag{37}$$

The distance Δ_ω, as defined in (6), can be considered as a function of ω and θ. We study the plot of Δ_ω for $e = 1, \theta \in [0, \pi)$ and $\omega \in [0, \frac{1}{2}]$, in Figure 6, and notice that for a fixed $\theta \in [0, \pi)$, $\omega = \frac{1}{2}$ is the maximum. For $\omega \in [\frac{1}{2}, 1]$ the graph is symmetric.

This leads to the next observation.

Observation 4.1.

$$\Delta_\omega < \Delta_{\frac{1}{2}} < \chi_{\frac{1}{2}} \theta e, \omega \in [0, 1]. \tag{38}$$

In view of the proofs of Theorems 3.4, 3.5 we conjecture,

Conjecture 4.2. *(Smoothness) Let $Subd^L$ be a converging linear subdivision scheme for points. Let $Subd^C$ be the modified $Subd^L$ for point-normal pairs based on the circle average. If the following conditions hold*

(i) $Subd^L$ is a converging scheme with positive coefficients $\{a_i\}$ in (37). It is easy to see that $Subd^L$ can be written in terms of repeated binary averages with positive weights.

(ii) the limit curves of $Subd^L$ are C^1,

(iii) $Subd^C$ converges,

then the limit curves of $Subd^C$ are C^1.

An important argument in the proofs of Theorem 3.4 and 3.5 is that the maximal angle θ^j converges to zero faster than the maximal edge length e^j, when $j \to \infty$. This is not guaranteed for an arbitrary modified subdivision scheme.

If the above relation holds, then a similar method of proof as in the proofs of Theorems 3.4, 3.5 can be used.

Otherwise, the following argument may be of help. The scheme for the normal is performed on the manifold of normals (unit circle), where each normal is represented by its angle with the $x-$axis, α_i^j. These angles are generated by $Subd^L$, and since its limits are C^1, the angles θ^j converges to zero at the rate $O(2^{-j})$. So, if e^j converges to zero even faster than θ^j, then probably the limit curves of $Subd^C$ are C^1.

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