Achievability Bounds for T-Fold Irregular Repetition Slotted ALOHA Scheme in the Gaussian Multiple Access Channel

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***Abstract*—We address the problem of massive random access for an uncoordinated Gaussian multiple access channel (MAC). The performance of T-fold irregular repetition slotted ALOHA (IRSA) scheme for this channel is investigated. The main difference of this scheme in comparison to IRSA is as follows: any collisions of order up to T can be resolved with some probability of error introduced by Gaussian noise. First, we generalize the density evolution method for T-fold IRSA and noisy channel and find optimal degree distributions for different values of T. Then we perform analysis and find minimal E_b/N_0 for a fixed length, rate and packet loss probability. The scheme is shown to work closer to finite length random coding bound proposed by Y. Polyanskiy, than existing solutions.**

I. Introduction

Existing wireless networks are designed with the goal of increasing a spectral efficiency in order to serve human users. Next generation of wireless networks will face a new challenge in the form of machine-type communication. Analysts predict that the number of devices connected to the network will exceed 50 millions by 2020. The main challenges are as follows: (a) huge number (billions) of autonomous devices connected to one access point, (b) low energy consumption, (c) short data packets. This problem has attracted attention of 3GPP standardization committee under the name of mMTC (massive machine-type communication).

Let us describe the system model. There are $K_{\text{tot}} \gg 1$ users, of which only K are active in each time instant. Communication proceeds in a frame-synchronized fashion (this can be implemented with use of beacons). The length of each frame is n. Each active user has k bits to transmit during a frame. The main goal is to minimize the energy-per-bit spent by each of the users. We are interested in grant-free access (5G terminology), i.e. active users transmit their data, without any resource requests.

This paper deals with construction of low-complexity random coding schemes for the Gaussian multiple-access channel (GMAC) with equal-power users, i.e.

$$\mathbf{y} = \sum_{i=1}^{K_{\text{tot}}} s_i \mathbf{x}_i + \mathbf{z}, \tag{1}$$

where $\mathbf{x}_i \in \mathbb{R}^n$ is a codeword transmitted by the i-th user, s_i is an activity indicator for the i-th user, i.e. $s_i = 1$ if the i-th user is active and $s_i = 0$ otherwise. $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ is an additive white Gaussian noise (AWGN). Following [1] we assume all the users to use the same message set $[M] \triangleq \{1, \dots, M\}$ and the same codebook $\mathcal{C} = \{\mathbf{x}(\omega)\}_{\omega=1}^{M}$ of size M. Let ω_i denote the message of the i-th user. To transmit the message ω_i the user will use a codeword $\mathbf{x}_i = \mathbf{x}(\omega_i)$. We require in addition that $||\mathbf{x}(\omega)||_2^2 \leq nP$, which means a natural power constraint.

Decoding is done up to permutation of messages. We only require the decoder to output a set $\mathcal{L}(\mathbf{y}) = (\omega_1, \omega_2, \dots, \omega_J) \in [M]^J$, $J \leq K$. Thus in accordance to [1] we decouple the user identification problem and the data transmission problem. The probability of error (per user) is defined as follows

$$P_e = \max_{|(s_1, s_2, \dots, s_{K_{\text{tot}}})| = K} \frac{1}{K} \sum_{i=1}^{K_{\text{tot}}} s_i \Pr(W_i \notin \mathcal{L}(\mathbf{y})).$$

Let us emphasize the main differences from the classical setting. Almost all well-known low-complexity coding solutions for the traditional MAC channel (e.g. [2]) assume coordination between the users. Due to the gigantic number users we assume them to be symmetric, i.e. the users use the same codes and equal powers.

We continue the line of work started in [1], [3], [4]. In [1] the bounds on the performance of finite-length codes for GMAC are presented. In [3] Ordentlich and Polyanskiy describe the first low-complexity coding paradigm for GMAC. The improvement (it terms of required E_b/N_0) was given in [4]. The overall scheme can be called T-fold irregular repetition slotted ALOHA (IRSA, [5], [6]) scheme for GMAC. The main difference of this scheme in comparison to IRSA is as follows: any collisions of order up to T can be resolved with some probability of error introduced by Gaussian noise.

In this paper we investigate the potential capabilities of T-fold IRSA. The authors of [4] suggested to split the error probability into three terms: interference cancellation (IC) error, preamble error and decoder error. Then a union bound was applied. We were able to write joint density evolution rules, which include all the terms. In what follows we show, that this change allows us to better predict the error probability. We also note the following thing. In [4] in each slot preambles were used to identify active users. So the packet consists of preamble and data part. The authors used finite length random coding bound [1] to estimate the probability of decoding failure in the data part. We emphasize, that preambles are not needed to characterize the potential capabilities (derive achievability bounds) of T-fold IRSA scheme as we do not need to identify users, that are active in a slot to use IC algorithm.

Our contribution is as follows. Achievability bounds for T-fold IRSA are derived. To achieve this goal we generalized the density evolution method for T-fold IRSA and noisy channel and find optimal degree distributions for different values of T. Then we perform analysis and find minimal E_b/N_0 for a fixed length, rate and packet loss probability. The scheme is shown to work closer to finite length random coding bound proposed by Y. Polyanskiy, than existing solutions.

II. Preliminaries

A. System model

The main features of the scheme are as follows:

- transmission is performed in a frame-synchronized fashion. The length of a frame is n channel uses;
- the frame consists of M slots of size $\tilde{n} = n/M$ channel uses;
- the user chooses a message ω, then encodes it and obtains a codeword $\mathbf{x}(\omega)$ of length $\tilde{n}$;
- users repeat their codewords in multiple slots. The distribution of the repetition count is denoted by $D[i]$. This distribution is the same for all the users;
- the number of repetitions r and the r slots in which to send are chosen based on the message ω (see [4]): as ω is distributed uniformly on $[M]$ the slots are chosen uniformly at random (definitely without repetitions) from M existing slots.

B. Interference cancellation decoder

Decoding algorithm is based on successive cancellation approach. The algorithm is iterative. The slot with the minimum collision order is selected at every step. If the order of collision is less or equal to T we resolve the collision. As a result some messages (in collision) are decoded successfully, some of the messages are decoded incorrectly. Then all successfully decoded messages are removed from other slots (if the message was transmitted by user more than once), the slot itself is marked as resolved and collision indices are updated for all other slots. We note, that we can always find where the replicas were transmitted as these positions are chosen based on the data (so in contrast to [5] we do not need to store the pointers).

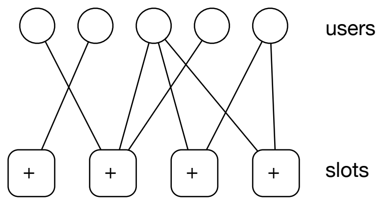

Fig. 1: Tanner graph representation

The algorithm stops when all the slots are either decoded or empty (see Algorithm 1 for more details).

Algorithm 1 Interference cancellation decoder

while slot set not empty **do**
  Remove empty slots
  Select slot with the minimal collision order t
  if $t \leq T$ **then**
    Resolve the collision
    Subtract decoded packets from all the slots
    Update collision indices for all slots
  end if
  Remove decoded slot from slot set
end while

Remark 1: We note, that during the slot decoding the errors may occur, i.e. some of the packets (codewords) may be decoded incorrectly. In what follows we assume, that we can always detect the error packets (the packets include control information).

C. Tanner graph representation

The transmission and decoding processes can be described with the use of a bipartite graph, which is called the Tanner graph [7]. The vertex set of the graph consists of the set of user nodes $V = \{v_1, v_2, \ldots, v_K\}$ which correspond to the set of users and the set of slot nodes $C = \{c_1, c_2, \ldots, c_M\}$ which correspond to signals received in slots. The user node v_i and the slot node c_j are connected with an edge if and only if the i-th users transmitted a packet in the j-th slot.

D. Degree distributions

Let $L(x) = \sum_i L_i x^i$ and $\lambda(x) = \sum_i \lambda_i x^{i-1}$ denote the user node degree distributions from node and edge degree perspective, respectively. We recall (see e.g. [8]), that L_i and λ_i denote respectively the fractions of user nodes of degree i and the fraction of edges incident to user nodes of degree i. Also recall, that $\lambda(x) = L'(x)/L'(1)$. In our case $L_i = D[i]$. Analogously, let $R(x) = \sum_i R_i x^i$ and $\rho(x) = \sum_i \rho_i x^{i-1}$ denote the slot node degree distributions from node and edge degree perspective, respectively.

Let $G = K/M$. Let us consider the j-th slot. Each user chooses this slot for transmission independently with probability $\frac{L'(1)}{M} = \frac{GL'(1)}{K}$. Thus, the slot node distribution (from node perspective) is $\text{Bino}\left(K, \frac{GL'(1)}{K}\right)$. In the limit

$K \to \infty$ this distribution becomes a Poisson distribution. In what follows we use $R(x) = \rho(x) = e^{-GL'(1)(1-x)}$.

III. Density evolution

Following [1] we want to minimize the required energy-per-bit E_b/N_0. For this purpose we suggest a density evolution method, which helps us to choose the system parameters.

Similar to [5], [6] we consider the ensemble of Tanner graphs $\mathcal{G}(K; M; c\lambda(x); \rho(x))$ corresponding to the multiple-access scheme with K users, M slots, and the degree distributions $\lambda(x)$ and $\rho(x)$. We are interested in the decoding performance averaged over the ensemble $\mathcal{G}(K; M; \lambda(x); \rho(x))$ in the limit as $K, M \to \infty$.

The major difference of our approach in comparison to [5], [6] is that we

- consider co-called T-fold IRSA, i.e. collisions of order up to T can be resolved in slot node with some probability of error introduced by Gaussian noise;
- take into account a noisy channel (AWGN channel to be precise);
- take into account finite length effect as the slots have small length;
- take into account a transmit energy. Assume we use a strategy with $L(x) = x^2$. In this case we spent 2 times more energy while transmitting in comparison to $L(x) = x$ strategy;

Let us fix the slot length $\tilde{n}$, the number of information bits k to be sent by each user, the maximal number of iterations ℓ, the average transmit power P (linear scale) and $L(x)$. Then the average energy per information bit can be calculated as follows

$$\frac{E_b}{N_0} = \frac{\tilde{n}PL'(1)}{2k},$$

we note, that $L'(1)$ is actually the average number of transmissions.

Let us introduce a notation. By $P_e(\tilde{n}, k, P, t)$ we denote the error probability per user, calculated with use of random coding bound [1]. Now let us write the density evolution rules. By x_l and y_l we denote the probability that an outgoing message from the user node and slot node, respectively, are erased during the l-th iteration. We start with initial condition $x_0 = 1$, which means, that the user messages are erased at the beginning and we observe only the noisy signal sums in slots.

$$\begin{aligned} y_{l+1} &= 1 - \sum_{r=1}^{r_{\max}} \rho_r \Big[\sum_{t=1}^{\min(r,T)-1} (1 - P_e(\tilde{n}, k, P, t)) \binom{r-1}{t} \\ &\times x_l^t (1 - x_l)^{r-1-t} \Big] \\ x_l &= \lambda(y_l), \ 1 \le l < \ell. \\ x_\ell &= L(y_\ell). \end{aligned}$$

Remark 2: We note, that

$$\lim_{\ell \to \infty} x_\ell > 0.$$

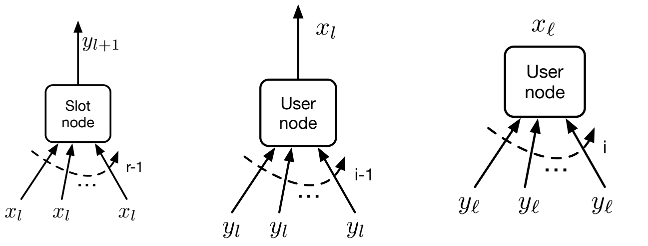

Fig. 2: Density evolution rules

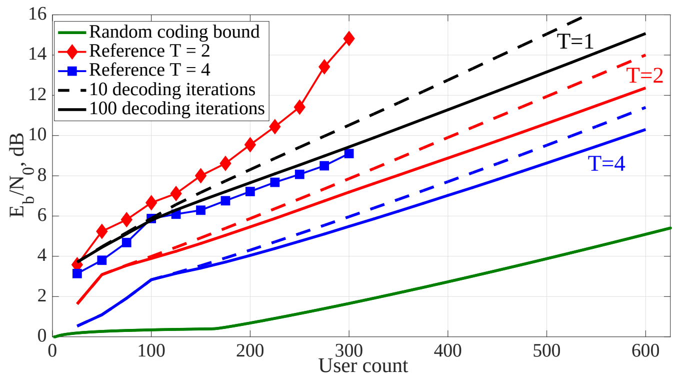

Fig. 3: Minimum E_b/N_0 required to achieve less than 5% error rate as a user count function

because of finite length effects in the slot. So in what follows we do not consider infinite number of iterations and fix ℓ.

Proof:

Consider the l-th iteration. Let us consider the slot node of degree r. We want to calculate the erasure probability of the outgoing message y_{l+1} based on incoming messages (with erasure probabilities x_l). The probability can be calculated as follows

$$1 - \sum_{t=1}^{\min(r,T)-1} (1 - P_e(\tilde{n}, k, P, t)) \binom{r-1}{t} x_l^t (1 - x_l)^{r-1-t}.$$

Each term in the sum corresponds to a probability of collision of order t. We are interested only in collisions of order $t \le T$. The probability of correct outgoing message decoding is given by $1 - P_e(\tilde{n}, k, P, t)$. We only need to apply the sum rule of total probability to obtain the needed result. Recall, that ρ_r is the probability, that the outgoing edge is connected to a slot node of degree r.

The rule in user node coincides with the rules from [5], [6]. ■

IV. Numerical results

We choose the same system parameters as in [1], [3], [4] for honest comparison.

Parameter	Description
$n = 3 \times 10^4$	Frame length
$k = 100$	Number of information bits
$P_e = 0.05$	Maximum error probability allowed

K	DE for $T = 2$, 100 iterations	M	E_b/N_0, dB
25	$1.0000x$	79	1.63
50	$0.4447x + 0.5553x^2$	61	3.09
100	$0.1261x + 0.8739x^2$	66	3.89
150	$0.1155x + 0.8845x^2$	97	4.64
200	$0.1226x + 0.8774x^2$	128	5.46
250	$0.1252x + 0.8748x^2$	159	6.32
300	$0.1591x + 0.6886x^2 + 0.1524x^3$	181	7.19
350	$0.1776x + 0.5895x^2 + 0.2329x^3$	205	8.03
400	$0.1860x + 0.5429x^2 + 0.2711x^3$	232	8.88
450	$0.2005x + 0.4861x^2 + 0.3135x^3$	258	9.73
500	$0.2048x + 0.4620x^2 + 0.3333x^3$	286	10.60
550	$0.2140x + 0.4269x^2 + 0.3590x^3$	312	11.48
600	$0.2218x + 0.3982x^2 + 0.3776x^3$ $+0.0015x^4 + 0.0005x^5$ $+0.000x^6 + 0.0001x^7$	339	12.37

K	DE for $T = 4$, 100 iterations	M	E_b/N_0, dB
25	$1.0000x$	25	0.53
50	$1.0000x$	41	1.09
100	$0.6260x + 0.3740x^2$	47	2.84
150	$0.3190x + 0.6810x^2$	43	3.41
200	$0.3228x + 0.6772x^2$	57	4.04
250	$0.3279x + 0.6721x^2$	71	4.75
300	$0.3259x + 0.6741x^2$	85	5.49
350	$0.3333x + 0.6667x^2$	99	6.24
400	$0.3298x + 0.6702x^2$	113	7.03
450	$0.3349x + 0.6651x^2$	127	7.82
500	$0.3381x + 0.6619x^2$	140	8.63
550	$0.3386x + 0.6436x^2 + 0.0177x^3$	153	9.46
600	$0.3445x + 0.6077x^2 + 0.0477x^3$	166	10.30

K	DE for $T = 1$, 100 iterations	M	E_b/N_0, dB
25	$0.0920x + 0.9080x^2$	68	3.71
50	$1.0000x^2$	106	4.45
100	$1.0000x^2$	195	5.79
150	$0.0009x + 0.4560x^2 + 0.5431x^3$	203	6.77
200	$0.1138x + 0.0931x^2 + 0.7930x^3$	251	7.66
250	$0.1371x + 0.0146x^2 + 0.8482x^3$	309	8.54
300	$0.1450x + 0.8550x^3$	368	9.43
350	$0.1466x + 0.8534x^3$	427	10.35
400	$0.1531x + 0.7928x^3 + 0.0540x^4$	482	11.28
450	$0.1610x + 0.7137x^3 + 0.1253x^4$	535	12.21
500	$0.1647x + 0.6666x^3 + 0.1687x^4$	590	13.16
600	$0.1700x + 0.6125x^3$ $+0.1955x^4 + 0.0219x^5$	700	15.07

A. Optimization procedure

The goal is to find the optimal slot count M and a polynomial $L(x)$ in order to minimize the E_b/N_0 under the maximum error probability allowed.

The optimization procedure is conducted separately for every user count and consists of two sub-procedures. The first one is a constrained local minimum search with respect to L coefficients and the number of slots used in the system. The constraints are formed by

- $L_i \geq 0 \quad \forall i = 1, \dots, L_{\max}$,
- $L(1) = 1$

The error probability is minimized under fixed E_b/N_0 at this step. As soon as constrained local minimum search procedures can search the local minimum only, one need to run multiple optimization procedures starting from different random initial points within constraints.

The second sub-procedure is to find a minimum E_b/N_0 that satisfies maximum allowed error probability P_e. We expect the error probability at the optimal configuration to be a monotonic function of E_b/N_0 and use a binary search.

The numerical experiment results show that $L(x)$ behaves smoothly when varying the number of users. This means that a global minimum is found at every optimization point. Note, that the error probability has multiple local minimums, because the slot count changes sharply at several points.

B. Simulation results

Interference cancellation algorithm was tested via Gaussian MAC Monte Carlo simulations. The result of a single run is a set of slots and the number of simultaneous transmissions (or collision index) for each slot. Each user selects the number of transmissions in accordance to $L(x)$ and then selects particular non-coinciding slots from uniform distribution during each run. The same E_b/N_0 is assumed for all slots.

The decoding is done in accordance to Algorithm 1. The only thing we need to explain is how we resolve the collisions. Error probability is set to 1 if the number of simultaneous transmissions within some slot exceeds the threshold ($T \in \{1, 2, 4\}$). If the order of collision is less or equal to T, then the error probability is calculated independently for each transmitted message in a slot in accordance to finite length random coding bound from [1].

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