A Proof of CSP Dichotomy Conjecture

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Abstract

Many natural combinatorial problems can be expressed as constraint satisfaction problems. This class of problems is known to be NP-complete in general, but certain restrictions on the form of the constraints can ensure tractability. The standard way to parameterize interesting subclasses of the constraint satisfaction problem is via finite constraint languages. The main problem is to classify those subclasses that are solvable in polynomial time and those that are NP-complete. It was conjectured that if a constraint language has a weak near unanimity polymorphism then the corresponding constraint satisfaction problem is tractable, otherwise it is NP-complete.

In the paper we present an algorithm that solves Constraint Satisfaction Problem in polynomial time for constraint languages having a weak near unanimity polymorphism, which proves the remaining part of the conjecture.

1 Introduction

Formally, the *Constraint Satisfaction Problem (CSP)* is defined as a triple $\langle \mathbf{X}, \mathbf{D}, \mathbf{C} \rangle$, where

- $\mathbf{X} = \{x_1, \dots, x_n\}$ is a set of variables,
- $\mathbf{D} = \{D_1, \dots, D_n\}$ is a set of the respective domains,
- $\mathbf{C} = \{C_1, \dots, C_m\}$ is a set of constraints,

where each variable x_i can take on values in the nonempty domain D_i, every *constraint* $C_j \in \mathbf{C}$ is a pair (t_j, ρ_j) where t_j is a tuple of variables of length m_j, called the *constraint scope*, and ρ_j is an m_j-ary relation on the corresponding domains, called the *constraint relation.*

The question is whether there exists *a solution* to $\langle \mathbf{X}, \mathbf{D}, \mathbf{C} \rangle$, that is a mapping that assigns a value from D_i to every variable x_i such that for each constraints C_j the image of the constraint scope is a member of the constraint relation.

In this paper we consider only CSP over finite domains. The general CSP is known to be NP-complete [16, 18]; however, certain restrictions on the allowed form of constraints involved may ensure tractability (solvability in polynomial time) [10, 13, 14, 15, 5, 9]. Below we provide a formalization to this idea.

To simplify the presentation we assume that all the domains $D_1, \dots, D_n$ are subsets of a finite set A. By R_A we denote the set of all finitary relations on A, that is, subsets of A^m for some m. Then all the constraint relations can be viewed as relations from R_A.

For a set of relations $\Gamma \subseteq R_A$ by $\mathrm{CSP}(\Gamma)$ we denote the Constraint Satisfaction Problem where all the constraint relations are from Γ. The set Γ is called *a constraint language.*

Another way to formalize the Constraint Satisfaction Problem is via conjunctive formulas. Every h-ary relation on A can be viewed as a predicate, that is, a mapping $A^h \to \{0,1\}$. Suppose $\Gamma \subseteq R_A$, then CSP(Γ) is the following decision problem: given a formula

$$\rho_1(x_{1,1},\ldots,x_{1,n_1}) \wedge \cdots \wedge \rho_s(x_{s,1},\ldots,x_{1,n_s})$$

where $\rho_i \in \Gamma$ for every i; decide whether this formula is satisfiable.

It is well known that many combinatorial problems can be expressed as CSP(Γ) for some constraint language Γ. Moreover, for some sets Γ the corresponding decision problem can be solved in polynomial time; while for others it is NP-complete. It was conjectured that CSP(Γ) is either in P, or NP-complete [11].

Conjecture 1. *Suppose $\Gamma \subseteq R_A$ is a finite set of relations. Then* CSP(Γ) *is either solvable in polynomial time, or NP-complete.*

We say that an operation $f\colon A^n \to A$ *preserves* the relation $\rho \in R_A$ of arity m if for any tuples $(a_{1,1},\ldots,a_{1,m}),\ldots,(a_{n,1},\ldots,a_{n,m}) \in \rho$ the tuple $(f(a_{1,1},\ldots,a_{n,1}),\ldots,f(a_{1,m},\ldots,a_{n,m}))$ is in ρ. We say that an operation *preserves a set of relations* Γ if it preserves every relation in Γ. A mapping $f : A \to A$ is called *an endomorphism* of Γ if it preserves Γ.

Theorem 1.1. *[5] Suppose $\Gamma \subseteq R_A$. If f is an endomorphism of Γ, then $CSP(\Gamma)$ is polynomially reducible to $CSP(f(\Gamma))$ and vice versa, where $f(\Gamma)$ is a constraint language with domain $f(\Gamma)$ defined by $f(\Gamma) = \{f(\rho)\colon \rho \in \Gamma\}$.*

A constraint language is *a core* if every endomorphism of Γ is a bijection. It is not hard to show that if f is an endomorphism of Γ with minimal range, then $f(\Gamma)$ is a core. Another important fact is that we can add all singleton unary relations to a core constraint language without increasing the complexity of its CSP. By $\sigma_{=a}$ we denote the unary relation $\{a\}$.

Theorem 1.2. *[5] Let $\Gamma \subseteq R_A$ be a core constraint language, and $\Gamma' = \Gamma \cup \{\sigma_{=a} \mid a \in A\}$, then $CSP(\Gamma')$ is polynomially reducible to $CSP(\Gamma)$.*

Therefore, to prove Conjecture 1 it is sufficient to consider only the case when Γ contains all unary singleton relations. In other words, all the predicates $x = a$, where $a \in A$, are in the constraint language Γ.

In [20] Schaefer classified all tractable constraint languages over two-element domain. In [7] Bulatov generalized the result for three-element domain. His dichotomy theorem was formulated in terms of a G-set. Later, the dichotomy conjecture was formulated in several different forms (see [5]).

The result of Mckenzie and Maróti [17] allows us to formulate the dichotomy conjecture in the following nice way. An operation f is called *a weak near-unanimity operation (WNU)* if $f(y,x,\ldots,x) = f(x,y,x,\ldots,x) = \cdots = f(x,x,\ldots,x,y)$. An operation f is called *idempotent* if $f(x,x,\ldots,x) = x$.

Conjecture 2. *Suppose $\Gamma \subseteq R_A$ is a finite set of relations. Then* CSP(Γ) *can be solved in polynomial time if there exists a WNU preserving Γ; $CSP(\Gamma)$ is NP-complete otherwise.*

It is not hard to see that the existence of a WNU preserving Γ is equivalent to the existence of a WNU preserving a core of Γ, and also equivalent to the existence of an idempotent WNU preserving the core. Hence, Theorems 1.1 and 1.2 imply that it is sufficient to prove Conjecture 2 for a core and an idempotent WNU.

One direction of this conjecture follows from [17].

Theorem 1.3. *[17] Suppose $\Gamma \subseteq R_A$, $\{\sigma_{=a} \mid a \in A\} \subseteq \Gamma$. If there exists no WNU preserving Γ, then* $\mathrm{CSP}(\Gamma)$ *is NP-complete.*

The dichotomy conjecture was proved for many special cases: for CSPs over undirected graphs [12], for CSPs over digraphs with no sources or sinks [2], for constraint languages containing all unary relations [6], and many others. Recently, a proof of the dichotomy conjecture was announced by Andrei Bulatov [8]. More information about the algebraic approach to CSP can be found in [3].

In this paper we present an algorithm that solves $\mathrm{CSP}(\Gamma)$ in polynomial time if Γ is preserved by an idempotent WNU, and therefore prove the dichotomy conjecture.

The paper is organized as follows. In Section 2 we give main definitions, in Section 3 we explain the algorithm. In Section 4 we prove a theorem that explains the main idea of the algorithm and formulate theorems that prove correctness of the algorithm. In Section 5 we give an example that explains how the algorithm works for a system of linear equations in $\mathbb{Z}_4$.

In the next section we give the remaining definitions. In Section 7 we present properties of absorbing, central, PC, and linear reductions. The important fact we prove in this section is that the restriction of some variables to absorbing subuniverses, centers, PC subuniverses, or linear subuniverses implies the corresponding restriction of other variables.

In Section 8 we prove the auxiliary statements: we show that minimal reductions preserve cycle-consistency and irreducibility, prove properties of the operator $\mathrm{Con}(\rho, x)$, explain how a linear variable can be added, show that previous reductions cannot harm, and prove the existence of a bridge.

In the last section we prove the main theorems of this paper formulated in Section 4. First, we explain the existence of a next reduction. Then we prove the existence of a linked connected component, and derive the main theorems from this fact.

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2 Definitions

A set of operations is called *a clone* if it is closed under composition and contains all projections. For a set of operations M by $\mathrm{Clo}(M)$ we denote the clone generated by M.

An idempotent WNU w is called *special* if $x \circ (x \circ y) = x \circ y$, where $x \circ y = w(x, \ldots, x, y)$. It is not hard to show that for any idempotent WNU w on a finite set there exists a special WNU $w' \in \mathrm{Clo}(w)$ (see Lemma 4.7 in [17]).

A relation $\rho \subseteq A_1 \times \cdots \times A_n$ is called *subdirect* if for every i the projection of ρ onto the i-th coordinate is A_i. For a relation ρ by $\mathrm{pr}_{i_1,\ldots,i_s}(\rho)$ we denote the projection of ρ onto the coordinates $i_1, \ldots, i_s$.

2.1 Algebras

An algebra is a pair $\mathbf{A} := (A; F)$, where A is a finite set, called *universe*, and F is a family of operations on A, called *basic operations of* $\mathbf{A}$. In the paper we always assume that we have a special WNU preserving all constraint relations. Therefore, every domain D can be viewed as an algebra $(D; w)$. By $\mathrm{Clo}(\mathbf{A})$ we denote the clone generated by all basic operations of $\mathbf{A}$.

An equivalence relation σ on the universe of an algebra $\mathbf{A}$ is called *a congruence* if it is preserved by every operation of the algebra. A congruence (an equivalence relation) is called *proper*, if it is not equal to the full relation $A \times A$. We use standard universal algebraic notions of term operation, subalgebra, factor algebra, product of algebras, see [4]. We say that a subalgebra $\mathbf{R} = (R; F_R)$ is *a subdirect subalgebra* of $\mathbf{A} \times \mathbf{B}$ if R is a subdirect relation in $A \times B$.

2.2 Polynomially complete algebras

An algebra $(A; F_A)$ is called *polynomially complete (PC)* if the clone generated by F_A and all constants on A is the clone of all operations on A.

2.3 Linear algebra

A finite algebra $(A; w_A)$ is called *linear* if it is isomorphic to $(\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}; x_1 + \ldots + x_n)$ for prime numbers $p_1, \ldots, p_s$. It is not hard to show that for every algebra $(B; w_B)$ there exists a minimal congruence σ, called *the minimal linear congruence*, such that $(B; w_B)/\sigma$ is linear.

2.4 Absorption

Let $\mathbf{B} = (B; F_B)$ be a subalgebra of $\mathbf{A} = (A; F_A)$. We say that B absorbs $\mathbf{A}$ if there exists $t \in \mathrm{Clo}(\mathbf{A})$ such that $t(B, B, \ldots, B, A, B, \ldots, B) \subseteq B$ for any position of A. In this case we also say that B is an absorbing subuniverse of $\mathbf{A}$. If the operation t can be chosen binary then we say that B is a binary absorbing subuniverse of $\mathbf{A}$.

2.5 Center

Suppose $\mathbf{A} = (A; w_A)$ is a finite algebra with a special WNU operation. $C \subseteq A$ is called a *center* if there exists an algebra $\mathbf{B} = (B; w_B)$ with a special WNU operation of the same arity and a subdirect subalgebra $(R; w_R)$ of $\mathbf{A} \times \mathbf{B}$ such that there is no binary absorbing subuniverse in $\mathbf{B}$ and $C = \{a \in A \mid \forall b \in B \colon (a, b) \in R\}$.

2.6 CSP instance

An instance of the constraint satisfaction problem is called *a CSP instance*. Sometimes we use the same letter for a CSP instance and for the set of all constraints of this instance. For a variable z by D_z we denote the domain of the variable z.

We say that $z_1 - C_1 - z_2 - \cdots - C_{l-1} - z_l$ is *a path* in Θ if z_i, z_{i+1} are in the scope of C_i for every i. We say that *a path* $z_1 - C_1 - z_2 - \ldots C_{l-1} - z_l$ *connects* b *and* c if there exists $a_i \in D_{z_i}$ for every i such that $a_1 = b$, $a_l = c$, and the projection of C_i onto z_i, z_{i+1} contains the tuple (a_i, a_{i+1}).

A CSP instance is called *cycle-consistent* if for every variable z and $a \in D_z$, any path starting and ending with z in Θ connects a and a. A CSP instance Θ is called *linked* if for every variable z appearing in Θ and every $a, b \in D_z$ there exists a path starting and ending with z in Θ that connects a and b.

Suppose $\mathbf{X}' \subseteq \mathbf{X}$. Then we can define a projection of Θ onto $\mathbf{X}'$, that is a CSP instance where variables are elements of $\mathbf{X}'$ and constraints are projections of the constraints of Θ onto $\mathbf{X}'$. We say that an instance Θ is *fragmented* if the set of variables $\mathbf{X}$ can be divided into 2 nonempty disjoint sets $\mathbf{X_1}$ and $\mathbf{X_2}$ such that the constraint scope of any constraint of Θ either has variables only from $\mathbf{X_1}$, or only from $\mathbf{X_2}$.

A CSP instance Θ is called *irreducible* if any instance Θ' such that every constraint of Θ' is a projection of a constraint from Θ on some set of variables is fragmented, linked, or its solution set is subdirect.

We say that a constraint $((y_1,\dots,y_t);\rho_1)$ is *weaker than* a constraint $((z_1,\dots,z_s);\rho_2)$ if $\{y_1,\dots,y_t\}\subseteq\{z_1,\dots,z_s\}$, $\rho_2(z_1,\dots,z_s)\to\rho_1(y_1,\dots,y_t)$, and $\rho_1(y_1,\dots,y_t)\not\to\rho_2(z_1,\dots,z_s)$.

Let $D_i'\subseteq D_i$ for every i. A constraint C of Θ is called *crucial in* $(D_1',\dots,D_n')$ if Θ has no solutions in $(D_1',\dots,D_n')$ but the replacement of $C\in\Theta$ by all weaker constraints gives an instance with a solution in $(D_1',\dots,D_n')$. A CSP instance Θ is called *crucial in* $(D_1',\dots,D_n')$ if every constraint of Θ is crucial in $(D_1',\dots,D_n')$.

Remark 1. *Suppose Θ has no solutions in $(D_1',\dots,D_n')$. Then we can replace constraints of Θ by all weaker constraints until we get a CSP instance that is crucial in $(D_1',\dots,D_n')$.*

3 Algorithm

3.1 Main part

Suppose we have a constraint language Γ_0 that is preserved by an idempotent WNU operation. As it was mentioned before, Γ_0 is also preserved by a special WNU operation w. Let k_0 be the maximal arity of the relations in Γ_0. By Γ we denote the set of all relations of arity at most k_0 that are preserved by w. Obviously, $\Gamma_0\subseteq\Gamma$, therefore CSP(Γ_0) can be reduced to CSP(Γ)

In this section we provide an algorithm that solves CSP(Γ) in polynomial time. Suppose we have a CSP instance $\Theta=\langle\mathbf{X},\mathbf{D},\mathbf{C}\rangle$, where $\mathbf{X}=\{x_1,\dots,x_n\}$ is a set of variables, $\mathbf{D}=\{D_1,\dots,D_n\}$ is a set of the respective domains, $\mathbf{C}=\{C_1,\dots,C_q\}$ is a set of constraints. Let the arity of the WNU w be equal to m.

The algorithm is recursive, the list of all possible recursive calls is given in the end of this subsection. One of the main recursive calls is the reduction of a subuniverse D_i to D_i' such that either Θ has a solution with $x_i\in D_i'$, or it has no solutions at all.

Step 1. *Check whether Θ is cycle-consistent. If not then we reduce a domain D_i for some i or state that there are no solutions.*

Step 2. *Check whether Θ is irreducible. If not then we reduce a domain D_i for some i or state that there are no solutions.*

Step 3. *Replace every constraint of Θ by all weaker constraints. Recursively calling the algorithm, check that the obtained instance has a solution with $x_i=b$ for every $i\in\{1,2,\dots,n\}$ and $b\in D_i$. If not, reduce D_i to the projection onto x_i of the solution set of the obtained instance.*

By Theorem 4.3 we cannot loose the only solution while doing the following two steps.

Step 4. *If D_i has a binary absorbing subuniverse $B_i\subsetneq D_i$ for some i, then we reduce D_i to B_i.*

Step 5. *If D_i has a center $C_i\subsetneq D_i$ for some i, then we reduce D_i to C_i.*

By Theorem 4.4 we can do the following step.

Step 6. *If there exists a congruence σ on D_i such that the algebra $(D_i;w)/\sigma$ is polynomially complete, then we reduce D_i to any equivalence class of σ.*

By Theorem 4.1, it remains to consider the case when for every domain D_i there exists a congruence σ_i on D_i such that $(D_i; w)/\sigma_i$ is linear, i.e. it is isomorphic to $(\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_l}; x_1 + \cdots + x_m)$ for prime numbers $p_1, \ldots, p_l$. Moreover, σ_i is proper if $|D_i| > 1$.

We denote D_i/σ_i by L_i. We define a new CSP instance Θ_L with domains $L_1, \ldots, L_n$. To every constraint $((x_{i_1}, \ldots, x_{i_s}); \rho) \in \Theta$ we assign a constraint $((x'_{i_1}, \ldots, x'_{i_s}); \rho')$, where $\rho' \subseteq L_{i_1} \times \cdots \times L_{i_s}$ and $(E_1, \ldots, E_s) \in \rho' \Leftrightarrow (E_1 \times \cdots \times E_s) \cap \rho \neq \varnothing$. The constraints of Θ_L are all constraints that are assigned to the constraints of Θ.

Since every relation on $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_l}$ preserved by $x_1 + \ldots + x_m$ is known to be a conjunction of linear equations, the instance Θ_L can be viewed as a system of linear equations in $\mathbb{Z}_p$ for different p. Note that all essential variables of every equation have the same domain.

Our general idea is to add some linear equations to Θ_L so that for any solution of Θ_L there exists the corresponding solution of Θ. We start with the empty set of equations Eq, which is a set of constraints on $L_1, \ldots, L_n$.

Step 7. *Put* $Eq := \varnothing$.

Step 8. *Solve the system of linear equations* $\Theta_L \cup Eq$ *and choose independent variables* $y_1, \ldots, y_k$. *If it has no solutions then* Θ *has no solutions. If it has just one solution, then, recursively calling the algorithm, solve the reduction of* Θ *to this solution. Either we get a solution of* Θ, *or* Θ *has no solutions.*

Then there exist $Z = \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k}$ and a linear mapping $\phi\colon Z \to L_1 \times \cdots \times L_n$ such that any solution of $\Theta_L \cup Eq$ can be obtained as $\phi(a_1, \ldots, a_k)$ for some $(a_1, \ldots, a_k) \in Z$.

Note that for any tuple $(a_1, \ldots, a_k) \in Z$ we can check recursively whether Θ has a solution in $\phi(a_1, \ldots, a_k)$. To do this, we just need to solve an easier CSP instance (on smaller domains). Similarly, we can check whether Θ has a solution in $\phi(a_1, \ldots, a_k)$ for every $(a_1, \ldots, a_k) \in \mathbb{Z}$. To do this, we just need to check the existence of a solution in $\phi(0, \ldots, 0, 1, 0, \ldots, 0)$ and $\phi(0, \ldots, 0)$ for any position of 1.

Step 9. *Check whether* Θ *has a solution in* $\phi(0, \ldots, 0)$. *If it has then stop the algorithm.*

Step 10. *Put* $\Theta' := \Theta$. *Iteratively remove from* Θ' *all constraints that are weaker than some other constraints of* Θ'.

Step 11. *For every constraint* $C \in \Theta'$

1. *Let* Ω *be obtained from* Θ' *by replacing the constraint* $C \in \Theta'$ *by all weaker constraints without dummy variables. Remove from* Ω *all constraints that are weaker than some other constraints of* Ω.

2. *If* Ω *has no solutions in* $\phi(a_1, \ldots, a_k)$ *for some* $(a_1, \ldots, a_k) \in Z$, *then put* $\Theta' := \Omega$. *Repeat Step 11.*

At this moment, the CSP instance Θ' has the following property. Θ' has no solutions in $\phi(b_1, \ldots, b_k)$ for some $(b_1, \ldots, b_k) \in Z$, but if we replace any constraint $C \in \Theta'$ by all weaker constraints, then we get an instance that has a solution in $\phi(a_1, \ldots, a_k)$ for every $(a_1, \ldots, a_k) \in Z$. Therefore, Θ' is crucial in $\phi(b_1, \ldots, b_k)$.

In the remaining steps we will find a new linear equation that can be added to Θ_L. Suppose V is an affine subspace of $\mathbb{Z}_p^h$ of dimension $h-1$, thus V is the solution set of a linear equation $c_1x_1 + \cdots + c_hx_h = c_0$. Then the coefficients $c_0, c_1, \ldots, c_h$ can be learned (up to a multiplicative constant) by $(p \cdot h + 1)$ queries of the form "$(a_1, \ldots, a_h) \in V$?" as follows. First, we need at most $(h+1)$ queries to find a tuple $(d_1, \ldots, d_h) \notin V$. Then, to find this equation it is sufficient to check for every a and every i whether the tuple $(d_1, \ldots, d_{i-1}, a, d_{i+1}, \ldots, d_h)$ satisfies this equation.

Step 12. *Suppose Θ' is not linked. For each i from 1 to k*

1. *Check that for every $(a_1, \dots, a_i) \in \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_i}$ there exist $(a_{i+1}, \dots, a_k) \in \mathbb{Z}_{q_{i+1}} \times \cdots \times \mathbb{Z}_{q_k}$ and a solution of Θ' in $\phi(a_1, \dots, a_k)$.*

2. *If yes, go to the next i.*

3. *If no, then find an equation $c_1y_1 + \cdots + c_iy_i = c_0$ such that for every $(a_1, \dots, a_i) \in \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_i}$ satisfying $c_1a_1 + \cdots + c_ia_i = c_0$ there exist $(a_{i+1}, \dots, a_k) \in \mathbb{Z}_{q_{i+1}} \times \cdots \times \mathbb{Z}_{q_k}$ and a solution of Θ' in $\phi(a_1, \dots, a_k)$.*

4. *Add the equation $c_1y_1 + \cdots + c_iy_i = c_0$ to Eq.*

5. *Go to Step 8.*

If Θ' is linked, then by Theorem 4.5 there exists a constraint $((x_{i_1}, \dots, x_{i_s}), \rho)$ in Θ' and a subuniverse σ of $\mathbf{D_{i_1}} \times \cdots \times \mathbf{D_{i_s}} \times \mathbb{Z}_\mathbf{p}$ such that the projection of σ onto the first s coordinates is bigger than ρ but the projection of $\sigma \cap (D_{i_1} \times \cdots \times D_{i_s} \times \{0\})$ onto the first s coordinates is equal to ρ. Then we add a new variable z with domain $\mathbb{Z}_p$ and replace $((x_{i_1}, \dots, x_{i_s}), \rho)$ by $((x_{i_1}, \dots, x_{i_s}, z), \sigma)$. We denote the obtained instance by Υ. Let L be the set of all tuples $(a_1, \dots, a_k, b) \in \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k} \times \mathbb{Z}_p$ such that Υ has a solution with $z = b$ in $\phi(a_1, \dots, a_k)$. We know that the projection of L onto the first n coordinates is a full relation. Therefore L is defined by one linear equation. If this equation is $z = b$ for some $b \neq 0$, then both Θ' and Θ have no solutions. Otherwise, we put $z = 0$ in this equation and get an equation that describes all $(a_1, \dots, a_k)$ such that Θ' has a solution in $\phi(a_1, \dots, a_k)$. It remains to find this equation.

Step 13. *Suppose Θ' is linked.*

1. *Find an equation $c_1y_1 + \cdots + c_ky_k = c_0$ such that for every $(a_1, \dots, a_k) \in (\mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_k})$ satisfying $c_1a_1 + \cdots + c_ka_k = c_0$ there exists a solution of Θ' in $\phi(a_1, \dots, a_k)$.*

2. *If the equation was not found then Θ has no solutions.*

3. *Add the equation $c_1a_1 + \cdots + c_ka_k = c_0$ to Eq.*

4. *Go to Step 8.*

Note that every time we reduce our domains, we get constraint relations that are still from Γ.

We have four types of recursive calls of the algorithm:

1. we reduce one domain D_i, for example to a binary absorbing subuniverse or to a center (Steps 1, 4, 5, 6).

2. we solve an instance that is not linked. In this case we divide the instance into the linked parts and solve each of them independently (Steps 2, 12).

3. we replace every constraint by all weaker constraints and solve an easier CSP instance (Step 3).

4. we reduce every domain D_i such that $|D_i| > 1$ (Steps 8, 9, 11, 13).

Lemma 4.2 states the depth of the recursive calls of type 3 is at most $|\Gamma|$. It is easy to see that the depth of the recursive calls of type 2 and 4 is at most $|A|$.

3.2 Remaining parts

In this section we explain Steps 1, 2, and 12 of the algorithm, which were not clarified in the previous section.

Provide cycle-consistency. To provide cycle-consistency it is sufficient to use constraint propagation providing (2,3)-consistency. Formally, it can be done in the following way. First, for every pair of variables (x_i, x_j) we consider the intersections of projections of all constraints onto these variables. The corresponding relation we denote by $\rho_{i,j}$. For every $i, j, k \in \{1, 2, \dots, n\}$ we replace $\rho_{i,j}$ by $\rho'_{i,j}$ where $\rho'_{i,j}(x, y) = \exists z\ \rho_{i,j}(x, y) \wedge \rho_{i,k}(x, z) \wedge \rho_{k,j}(z, y)$. It is not hard to see that this replacement does not change the solution set.

We repeat this procedure while we can change some $\rho_{i,j}$. If at some moment we get a relation $\rho_{i,j}$ that is not subdirect in $D_i \times D_j$, then we can either reduce D_i or D_j, or, if $\rho_{i,j}$ is empty, state that there are no solutions. If we cannot change any relation $\rho_{i,j}$ and every $\rho_{i,j}$ is subdirect in $D_i \times D_j$, then the original CSP instance is cycle-consistent.

Solve the instance that is not linked. Suppose the instance Θ is not linked and not fragmented, then it can be solved in the following way. We say that an element $d_i \in D_i$ and an element $d_j \in D_j$ are *linked* if there exists a path that connects d_i and d_j. Let P be the set of pairs $(i; a)$ such that $i \in \{1, 2, \dots, n\}$, $a \in D_i$. Then P can be divided into the linked components.

It is easy to see that it is sufficient to solve the problem for every linked component and join the results. Precisely, for a linked component by D'_i we denote the set of all elements d such that (i, d) is in the component. It is easy to see that $\varnothing \subsetneq D'_i \subsetneq D_i$ for every i. Therefore, the reduction to $(D'_1, \dots, D'_n)$ is a CSP instance on smaller domains.

Check irreducibility. For every $k \in \{1, 2, \dots, n\}$ and every maximal congruence σ_k on D_k we do the following.

1. Put $I = \{k\}$.

2. Choose a constraint C having the variable x_i in the scope for some $i \in I$, choose another variable x_j from the scope such that $j \notin I$.

3. Denote the projection of C onto (x_i, x_j) by δ.

4. Put $\sigma_j(x, y) = \exists x' \exists y' \delta(x', x) \wedge \delta(y', y) \wedge \sigma_i(x', y')$. If σ_j is a proper equivalence relation, then add j to I.

5. go to the next C, x_i, and x_j in 2.

As a result we get a set I and a congruence σ_i on D_i for every $i \in I$. Put $\mathbf{X}' = \{x_i \mid i \in I\}$. It follows from the construction that for every equivalence class E_k of σ_k and every $i \in I$ there exists a unique equivalence class E_i of σ_i such that there can be a solution with $x_k \in E_k$ and $x_i \in E_i$. Thus, for every equivalence class of σ_k we have a reduction to the instance on smaller domains. Then for every i and $a \in E_i$ we consider the corresponding reduction and check whether there exists a solution with $x_i = a$.

Thus, we can check whether the solution set of the projection of the instance onto $\mathbf{X}'$ is subdirect or empty. If it is empty then we state that there are no solutions. If it is not subdirect, then we can reduce the corresponding domain. If it is subdirect, then we go to the next $k \in \{1, 2, \dots, n\}$ and next maximal congruence σ_k on D_k, and repeat the procedure.

4 Correctness of the Algorithm

4.1 Rosenberg completeness theorem

The main idea of the algorithm is based on a beautiful result obtained by Ivo Rosenberg in 1970, who found all maximal clones on a finite set. Applying this result to the clone generated by a WNU together with all constant operations, we can show that every algebra with a WNU operation has a binary absorption, a center, or it is polynomially complete or linear modular some congruence.

Theorem 4.1. *Suppose* $\mathbf{A} = (A; w)$ *is an algebra,* w *is a special WNU of arity* m*. Then one of the following conditions hold*

1. *there exists a binary absorbing set* $B \subsetneq A$*,*
2. *there exists a center* $C \subsetneq A$*,*
3. *there exists a proper congruence* σ *on* A *such that* $(A; w)/\sigma$ *is polynomially complete,*
4. *there exists a proper congruence* σ *on* A *such that* $(A; w)/\sigma$ *is isomorphic to* $(\mathbb{Z}_p; x_1 + \cdots + x_m)$*.*

Proof. Let us prove this statement by induction on the size of A. If we have a binary absorbing subuniverse in A then there is nothing to prove. Let M be the clone generated by w and all constant operations on A. If M is the clone of all operations, then $(A; w)$ is polynomially complete.

Otherwise, by Rosenberg's Theorem [19], M belongs to one of the following maximal clones.

1. Maximal clone of monotone operations, that is, a clone of operations preserving a partial order relation with the greatest and the least element;
2. Maximal clone of autodual operations, that is, a clone of operations preserving the graph of a permutation of a prime order without a fixed element;
3. Maximal clone defined by an equivalence relation;
4. Maximal clone of quasi-linear operations;
5. Maximal clone defined by a central relation;
6. Maximal clone defined by an h-universal relation.

Let us consider all the cases.

1. The least element of the partial order can be viewed as a center. Since there is no binary absorbing subuniverse, we have a center in A.
2. Constants are not autodual operations. This case cannot happen.
3. Let δ be a maximal congruence on $\mathbf{A}$. We consider a factor algebra $(A; w)/\delta$ and apply the inductive assumption.
   (a) If $\mathbf{A}/\delta$ has a binary absorbing subuniverse $B' \subseteq A/\delta$, then we can check that $\bigcup_{E \in B'} E$ is a binary absorbing subuniverse of A.
   (b) If $\mathbf{A}/\delta$ has a center $C' \subseteq A/\sigma$, then we can check that $\bigcup_{E \in C'} E$ is a center of A.

(c) Suppose $(\mathbf{A}/\delta)/\sigma$ is polynomially complete. Since δ is a maximal congruence, σ is the equality relation and $\mathbf{A}/\delta$ is polynomially complete.

(d) Suppose $(\mathbf{A}/\delta)/\sigma$ is isomorphic to $(\mathbb{Z}_p; x_1 + \dots + x_m)$. Since δ is a maximal congruence, σ is the equality relation and $\mathbf{A}/\delta$ is isomorphic to $(\mathbb{Z}_p; x_1 + \dots + x_m)$.

4. By Lemma 6.4 from [21], we know that $w(x_1, \dots, x_m) = x_1 + \dots + x_m$, where $+$ is the operation in an abelian group. We assume that $\mathbf{A}$ has no nontrivial congruences, otherwise we refer to case 3. Then the algebra $\mathbf{A}$ is simple and isomorphic to $(\mathbb{Z}_p; x_1 + \dots + x_m)$ for a prime number p.

5. We consider the central relation ρ. Let k be the arity of ρ. It is not hard to see that the existence of a binary absorbing subuniverse on $\underbrace{\mathbf{A} \times \dots \times \mathbf{A}}_{k-1}$ implies the existence of a binary absorbing subuniverse on $\mathbf{A}$ (see Lemma 7.3). Therefore, the center of ρ can be viewed as a center.

6. By Corollary 5.10 from [21] this case cannot happen.

□

4.2 Correctness of the algorithm

Lemma 4.2. *The depth of the recursive calls of type 3 in the algorithm is less than $|\Gamma|$.*

Proof. First, we introduce a partial order on the set of relations in Γ in the following way. We say that $\rho_1 \leqslant \rho_2$ if one of the following conditions hold

1. the arity of ρ_1 is less than the arity of ρ_2.

2. the arity of ρ_1 equals the arity of ρ_2, $\mathrm{pr}_i(\rho_1) \subseteq \mathrm{pr}_i(\rho_2)$ for every i, $\mathrm{pr}_j(\rho_1) \neq \mathrm{pr}_j(\rho_2)$ for some j.

3. the arity of ρ_1 equals the arity of ρ_2, $\mathrm{pr}_i(\rho_1) = \mathrm{pr}_i(\rho_2)$ for every i, and $\rho_1 \supseteq \rho_2$.

It is easy to see that any reduction makes every relation smaller or does not change it. Since our constraint language Γ is finite, there can be at most $|\Gamma|$ recursive calls of type 3. □

The following three theorems will be proved in Section 9.

Theorem 4.3. *Suppose Θ is a cycle-consistent irreducible CSP instance, B is a binary absorbing set or a center of D_i. Then Θ has a solution if and only if Θ has a solution with $x_i \in B$.*

Theorem 4.4. *Suppose Θ is a cycle-consistent irreducible CSP instance, there does not exist a binary absorbing subuniverse or a center on D_j for every j, $(D_i; w)/\sigma$ is a polynomially complete algebra, E is an equivalence class of σ. Then Θ has a solution if and only if Θ has a solution with $x_i \in E$.*

Theorem 4.5. *Suppose the following conditions hold:*

1. Θ is a linked cycle-consistent irreducible CSP instance with domain set $(D_1, \dots, D_n)$;

2. there does not exist a binary absorbing subuniverse or a center on D_j for every j;

3. if we replace every constraint of Θ by all weaker constraints then the obtained instance has a solution with $x_i = b$ for every i and $b \in D_i$.

4. Θ_L is Θ factorized by the minimal linear congruences;

5. $(D'_1, \dots, D'_n)$ is a solution of Θ_L, and Θ is crucial in $(D'_1, \dots, D'_n)$.

Then there exists a constraint $((x_{i_1}, \dots, x_{i_s}), \rho)$ in Θ and a subuniverse ζ of $\mathbf{D_{i_1}} \times \cdots \times \mathbf{D_{i_s}} \times \mathbb{Z}_\mathbf{p}$ such that the projection of ζ onto the first s coordinates is bigger than ρ but the projection of $\zeta \cap (D_{i_1} \times \cdots \times D_{i_s} \times \{0\})$ onto the first s coordinates is equal to ρ.

5 An example in $\mathbb{Z}_4$

In this section we demonstrate the main part of the algorithm for a system of linear equations in $\mathbb{Z}_4$. Suppose we have a system

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + 2x_3 + 2x_4 = 0 \end{cases} \tag{1}$$

The minimal congruence σ such that $(\mathbb{Z}_4; x_1 + \ldots + x_5)/\sigma$ is linear is an equivalence relation modulo 2.

We write the corresponding system of linear equations in $\mathbb{Z}_2$, where $x'_i = x_i \mod 2$.

$$\begin{cases} x'_1 + x'_3 + x'_4 = 0 \\ x'_2 + x'_3 + x'_4 = 0 \\ x'_1 + x'_2 = 0 \end{cases} \tag{2}$$

We choose independent variables x'_1 and x'_3, and write the general solution: $x'_1 = x'_1, x'_2 = x'_1, x'_3 = x'_3, x'_4 = x'_1 + x'_3$. We check that (1) does not have a solution, corresponding to $x'_1 = x'_3 = 0$. Let us remove the last equation from (1).

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + x_2 = 2 \end{cases} \tag{3}$$

We check that (3) still has no solutions corresponding to $x'_1 = x'_3 = 0$.

We check that if we remove any equation from (3), then for any $a_1, a_3 \in \mathbb{Z}_2$ there will be a solution corresponding to $x'_1 = a_1$ and $x'_3 = a_3$. Hence we need to add exactly one equation to describe all pairs (a_1, a_3) such that (3) has a solution corresponding to $x'_1 = a_1$ and $x'_3 = a_3$. Let the equation be $c_1 x'_1 + c_3 x'_3 = c_0$. We need to find c_1, c_3, and c_0.

Since (3) has a solution corresponding to $x'_1 = 1, x'_3 = 0$, but no solutions for $x'_1 = 0, x'_3 = 1$, the equation is $x'_1 = 1$.

We add this equation to (2) and solve the new system of linear equations in $\mathbb{Z}_2$.

$$\begin{cases} x'_1 + x'_3 + x'_4 = 0 \\ x'_2 + x'_3 + x'_4 = 0 \\ x'_1 + x'_2 = 0 \\ x'_1 = 1 \end{cases} \tag{4}$$

The general solution of this system is $x'_1 = 1$, $x'_2 = 1$, $x'_3 = x'_3$, $x'_4 = x'_3 + 1$, where x'_3 is an independent variable. We go back to (1), and check whether it has a solution corresponding to $x'_3 = 0$. Thus, we find a solution $(1, 1, 0, 1)$.

While solving the system of equations, we just solved systems of linear equations in the field $\mathbb{Z}_2$ and constraint satisfaction problems on 2 element set (which are also equivalent to systems of linear equations in $\mathbb{Z}_2$).

6 The Remaining Definitions

6.1 Additional notations

We say that the i-th variable of a relation ρ is *compatible with the congruence* σ if $(a_1, \dots, a_n) \in \rho$ and $(a_i, b_i) \in \sigma$ implies $(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n) \in \rho$. We say that a relation is *compatible* with σ if every variable of this relation is compatible with σ.

We say that a relation ρ' *is obtained from* ρ *by factorization of the i-th variable by a congruence* σ if $\rho \subseteq A_1 \times \dots \times A_n$, $\rho' \subseteq A_1 \times \dots \times A_{i-1} \times A_i/\sigma \times A_{i+1} \times \dots \times A_n$, and

$$(a_1, \dots, a_{i-1}, E, a_{i+1}, \dots, a_n) \in \rho' \Leftrightarrow \exists a_i \in E \colon (a_1, \dots, a_n) \in \rho.$$

We say that a congruence σ is *irreducible* if it cannot be represented as an intersection of other binary relations $\delta_1, \dots, \delta_s$ compatible with σ. For an irreducible congruence σ on a set A by σ^* we denote the minimal binary relation $\delta \supsetneq \sigma$ compatible with σ.

For a relation ρ by $\mathrm{Con}(\rho, i)$ we denote the binary relation $\sigma(y, y')$ defined by

$$\exists x_1 \dots \exists x_{i-1} \exists x_{i+1} \dots \exists x_n \; \rho(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \wedge \rho(x_1, \dots, x_{i-1}, y', x_{i+1}, \dots, x_n).$$

For a constraint $C = \rho(x_1, \dots, x_n)$, by $\mathrm{Con}(C, x_i)$ we denote $\mathrm{Con}(\rho, i)$. For a set of constraints Ω by $\mathrm{Con}(\Omega, x)$ we denote the set $\{\mathrm{Con}(C, x) \mid C \in \Omega\}$.

For an algebra $\mathbf{A}$ by $\mathrm{ConPC}(\mathbf{A})$ we denote the intersection of all congruences σ such that $\mathbf{A}/\sigma$ is a PC algebra. A subuniverse A' of $\mathbf{A}$ is called a *PC subuniverse* if $A' = E_1 \cap \dots \cap E_s$, where E_i is an equivalence class of a congruence σ_i such that $\mathbf{A}/\sigma_i$ is a PC algebra. A variable is called *a PC variable* if its domain is a PC algebra.

For an algebra $\mathbf{A}$ by $\mathrm{ConLin}(\mathbf{A})$ we denote the minimal linear congruence. A subuniverse of $\mathbf{A}$ is called a *linear subuniverse* if it is compatible with $\mathrm{ConLin}(\mathbf{A})$.

6.2 Variety of algebras

We consider the variety of all algebras $\mathbf{A} = (A; w)$ such that w is a special WNU operation of arity m. In this paper every algebra and every domain is considered as an algebra in this variety. Every relation $\rho \subseteq A_1 \times \dots \times A_n$ appearing in this paper is a subalgebra of $\mathbf{A}_1 \times \dots \times \mathbf{A}_n$ for some algebras $\mathbf{A}_1, \dots, \mathbf{A}_n$ of this variety.

6.3 Formula, pp-formula, subconstraint

Every variable x appearing in the paper has its domain, which we denote by D_x.

A set of constraints is called a *formula*. Sometimes we write a formula as $C_1 \wedge \dots \wedge C_n$. For example, a CSP instance can be viewed as a formula. We say that a formula is *a tree-formula* if every there is no a path $z_1 - C_1 - z_2 - \dots C_{l-1} - z_l$ such that $l \geqslant 3$, $z_1 = z_l$, and all the constraints $C_1, \dots, C_{l-1}$ are different.

For a CSP instance Θ and a formula $\Omega \subseteq \Theta$ an expression $\Omega(x_1, \dots, x_n)$ is called a *pp-formula.* A pp-formula $\Omega(x_1, \dots, x_n)$ is called a *subconstraint of* Θ if Ω and $\Theta \setminus \Omega$ do not have common variables except for $x_1, \dots, x_n$.

We say that a pp-formula $\Omega(x_1, \dots, x_n)$ defines a relation ρ if $\rho(x_1, \dots, x_n) = \exists y_1 \dots \exists y_s\, \Omega$, where $\{x_1, \dots, x_n, y_1, \dots, y_s\}$ is the set of all variables appearing in Ω.

For a formula Ω by $\mathrm{Var}(\Omega)$ we denote the set of all variables of Ω. For a formula Ω and two sets of variables $x_1, \dots, x_n$ and $y_1, \dots, y_n$ by $\Omega^{y_1,\dots,y_n}_{x_1,\dots,x_n}$ we denote the formula obtained from Ω by replacement of every variable x_i by y_i.

For a formula Ω by $\mathrm{ExpCov}(\Omega)$ (*Expanded Coverings*) we denote the set of all formulas Ω' such that there exists a mapping $S : \mathrm{Var}(\Omega') \to \mathrm{Var}(\Omega)$ satisfying the following conditions:

1. for every constraint $(\rho; (x_1, \dots, x_n))$ of Ω' either the variables $S(x_1), \dots, S(x_n)$ are different and the constraint $(\rho; (S(x_1), \dots, S(x_n)))$ is weaker than or equal to some constraint of Ω, or ρ is a binary reflexive relation and $S(x_1) = S(x_2)$;

2. if a variable x appears in Ω and Ω' then $S(x) = x$.

If instead of item 1 we require that $(\rho; (S(x_1), \dots, S(x_n)))$ is a constraint of Ω, we define the set of formulas $\mathrm{Coverings}(\Omega)$. For a variable x we say that $S(x)$ is *the parent of* x.

Lemma 6.1. *Suppose* Θ *is a cycle-consistent irreducible CSP instance,* $\Theta' \in \mathrm{ExpCov}(\Theta)$. *Then* Θ' *is cycle-consistent and irreducible.*

Proof. For every path in Θ' there exists a corresponding path in Θ. Therefore Θ' is cycle-consistent. Assume that Θ' is not irreducible. Then there exists an instance Ω' consisting of projections of constraints from Θ' that is not linked, not fragmented, and its solution set is not subdirect. By Ω we denote the set of projections of constraints from Θ corresponding to the constraints of Ω' (corresponding constraints should have the same arity). Let us show that Ω is not linked. Assume the contrary. For any path in Ω connecting elements a and b of D_x we build a path connecting a and b in Ω' in the following way. We replace every constraint of Ω by the corresponding constraint of Ω', and glue them with any path in Ω' starting and ending with the corresponding variables having the same parent. Since Ω' is not fragmented, we can always do this. Since Ω is cycle-consistent, the obtained path connects a and b in Ω'. Thus, Ω is not linked, not fragmented, and its solution set is not subdirect, which contradicts the fact that Θ is irreducible. □

For a formula Θ and a variable x of this formula by $\mathrm{LinkedCon}(\Theta, x)$ we denote the congruence on the set D_x defined as follows: $(a, b) \in \mathrm{LinkedCon}(\Theta, x)$ if there exists a path in Θ that connects a and b.

6.4 Critical, key relations, and parallelogram property

We say that a relation ρ *has the parallelogram property* if any permutation of its variables gives a relation ρ' satisfying

$$\forall \alpha_1, \beta_1, \alpha_2, \beta_2 \colon (\alpha_1\beta_2, \beta_1\alpha_2, \beta_1\beta_2 \in \rho' \Rightarrow \alpha_1\alpha_2 \in \rho').$$

We say that *the i-th variable of a relation ρ is rectangular*, if for every $(a_i, b_i) \in \mathrm{Con}(\rho, i)$ and $(a_1, \dots, a_n) \in \rho$ we have $(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n) \in \rho$. We say that a relation is *rectangular* if all of its variables are rectangular. The following facts can be easily seen: if the i-th variable of ρ is rectangular then $\mathrm{Con}(\rho, i)$ is a congruence; if a relation has the parallelogram property then it is rectangular.

A relation $\rho \in R_A^h$ is called *essential* if it cannot be represented as a conjunction of relations with smaller arities. It is easy to see that any relation ρ can be represented as a conjunction of essential relations.

A relation $\rho \subseteq A_1 \times \cdots \times A_n$ is called *critical* if it cannot be represented as an intersection of other subalgebras of $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ and it has no dummy variables. For a critical relation ρ the minimal relation ρ' (a subalgebra of $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$) such that $\rho' \supsetneq \rho$ is called *the cover*.

Suppose $\rho \subseteq A_1 \times \cdots \times A_h$. A tuple $\Psi = (\psi_1, \psi_2, \ldots, \psi_h)$, where $\psi_i : A_i \to A_i$, is called a *unary vector-function*. We say that Ψ *preserves* ρ if $\Psi\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_h \end{pmatrix} := \begin{pmatrix} \psi_1(a_1) \\ \psi_2(a_2) \\ \vdots \\ \psi_h(a_h) \end{pmatrix} \in \rho$ for every $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_h \end{pmatrix} \in \rho$. We say that ρ is *a key relation* if there exists a tuple $\beta \in (A_1 \times \cdots \times A_h) \setminus \rho$ such that for every $\alpha \in (A_1 \times \cdots \times A_h) \setminus \rho$ there exists a vector-function Ψ which preserves ρ and gives $\Psi(\alpha) = \beta$. A tuple β is called a *key tuple* for ρ.

A constraint is called *critical/essential/key* if the constraint relation is critical/essential/key.

6.5 Reductions

A CSP instance is called *1-consistent* if every constraint of the instance is subdirect.

Suppose the domain set of the instance Θ is $D = (D_1, \ldots, D_n)$. The domain set $D' = (D'_1, \ldots, D'_n)$ is called *a reduction* if D'_i is a subuniverse of D_i for every i.

The reduction $D' = (D'_1, \ldots, D'_n)$ is called *1-consistent* if the instance obtained after reduction of every domain is 1-consistent.

We say that D' is *an absorbing reduction*, if there exists a term operation t such that D'_i is a binary absorbing subuniverse of D_i with the term operation t for every i. We say that D' is *a central reduction*, if D'_i is a center of D_i for every i. We say that D' is *a PC/linear reduction*, if D'_i is a PC/linear subuniverse of D_i and D_i does not have a center or binary absorbing subuniverse for every i. Additionally, we say that D' is *a minimal central/PC/linear reduction* if D' is a minimal center/PC/linear subuniverse of D_i for every i. We say that D' is *a minimal absorbing reduction* for a term operation t if D' is a minimal absorbing subuniverse of D_i with t for every i.

A reduction is called *nonlinear* if it is an absorbing, central, or PC reduction. A reduction D' is called *proper* if it is an absorbing, central, PC, or linear reduction such that $D' \neq D$.

We usually denote reductions by $D^{(j)}$ for some j (or by $D^{(\top)}$). In this case by $C^{(j)}$ we denote the constraint obtained after the reduction of the constraint C. Similarly, by $\Theta^{(j)}$ we denote the instance obtained after the reduction of Θ. For a relation ρ by $\rho^{(j)}$ we denote the relation ρ restricted to the corresponding domains of $D^{(j)}$. We sometimes say *factorization by* $(j+1)$ instead of factorization by $\mathrm{ConLin}(D_x^{(j)})$ or $\mathrm{ConPC}(D_x^{(j)})$ if $D^{(j+1)}$ is a PC or linear reduction. Sometimes we write $(a_1, \ldots, a_n) \in D^{(j)}$ to say that every a_i belongs to the corresponding $D_x^{(j)}$.

A *strategy* for a CSP instance Θ with a domain set D is a sequence of reductions $D^{(0)}, \ldots, D^{(s)}$, where $D^{(i)} = (D_1^{(i)}, \ldots, D_n^{(i)})$, such that $D^{(0)} = D$ and $D^{(i)}$ is a proper 1-consistent reduction of $\Theta^{(i-1)}$ for every $i \geqslant 1$. A strategy is called *minimal* if every reduction in the sequence is minimal.

6.6 Bridges

Suppose σ_1 and σ_2 are congruences on D_1 and D_2, correspondingly. A relation $\rho \subseteq D_1^2 \times D_2^2$ is called *a bridge* from σ_1 to σ_2 if the first two variables of ρ are compatible with σ_1, the last two variables of ρ are compatible with σ_2, $\mathrm{pr}_{1,2}(\rho) \supsetneq \sigma_1$, $\mathrm{pr}_{3,4}(\rho) \supsetneq \sigma_2$, and $(a_1, a_2, a_3, a_4) \in \rho$ implies

$$(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2.$$

Suppose σ_1, σ_2, σ_3 are irreducible congruences, we have a bridge ρ_1 from σ_1 to σ_2 and a bridge ρ_2 from σ_2 to σ_3. Then we can compose these bridges to define a bridge from σ_1 to σ_3, that is, we define the new bridge by $\exists y_1 \exists y_2 \rho_1(x_1, x_2, y_1, y_2) \wedge \rho_2(y_1, y_2, z_1, z_2)$.

A bridge $\rho \subseteq D^4$ is called *reflexive* if $(a, a, a, a) \in \rho$ for every $a \in D$.

We say that two congruences σ_1 and σ_2 on a set D are *adjacent* if there exists a reflexive bridge from σ_1 to σ_2.

Remark 2. *Since we can always put $\rho(x_1, x_2, x_3, x_4) = \sigma(x_1, x_3) \wedge \sigma(x_2, x_4)$, any congruence σ is adjacent with itself.*

A reflexive bridge ρ from σ_1 to σ_2 is called *optimal* if there does not exist a reflexive bridge ρ' from σ_1 to σ_2 such that $\rho'(x, x, y, y)$ is weaker than $\rho(x, x, y, y)$. Since we can compose two reflexive bridges together, the relation $\rho(x, x, y, y)$ is a congruence for any optimal bridge ρ. For an irreducible congruence σ by $\mathrm{Opt}(\sigma)$ we denote the congruence defined by $\rho(x, x, y, y)$ for an optimal bridge ρ from σ to σ. For a set of irreducible congruences Ω put $\mathrm{Opt}(\Omega) = \{\mathrm{Opt}(\sigma) \mid \sigma \in \Omega\}$.

We say that two constraints C_1 and C_2 are *adjacent* in a common variable x if $\mathrm{Con}(C_1, x)$ and $\mathrm{Con}(C_2, x)$ are adjacent. A formula is called *connected* if every constraint in the formula is rectangular and for every two constraints there exists a path that connects them. It can be shown (see Corollary 8.15.1) that every two constraints with common variable in a connected instance are adjacent.

6.7 Dummy variables

To simplify explanation and avoid collisions in this paper we assume that:

1. every time we replace a constraint by all weaker constraints, the weaker constraints have no dummy variables but might have smaller scopes;

2. if every constraint of Ω has no dummy variables, then every constraint of an instance $\Omega' \in \mathrm{ExpCov}(\Omega)$ has no dummy variables;

3. every constraint in a crucial instance has no dummy variables.

7 Absorption, Center, PC Congruence, and Linear Congruence

7.1 Binary Absorption

Lemma 7.1. *[1] Suppose ρ is defined by a pp-formula $\Omega(x_1, \ldots, x_n)$, Ω' is obtained from Ω by replacement of some constraint relations $\sigma_1, \ldots, \sigma_s$ by constraint relations $\sigma'_1, \ldots, \sigma'_s$ such that σ'_i absorbs σ_i with a term operation t for every i. Then the relation defined by $\Omega'(x_1, \ldots, x_n)$ absorbs ρ with the term operation t.*

Corollary 7.1.1. *Suppose $\rho \subseteq A_1 \times \cdots \times A_n$ is a relation such that $\mathrm{pr}_1(\rho) = A_1$, $C = \mathrm{pr}_1((C_1 \times \cdots \times C_n) \cap \rho)$, where C_i is an absorbing subuniverse in A_i with a term operation t for every i. Then C is an absorbing subuniverse in A_1 with the term operation t.*

Lemma 7.2. *Suppose $\kappa_A \subseteq A \times A$ is the equality relation, $\sigma \supseteq \kappa_A$, ω is a binary absorbing subuniverse in σ. Then $\omega \cap \kappa_A \neq \varnothing$.*

Proof. We prove by induction on the size of A. Suppose ω absorbs σ with a binary absorbing term operation t.

Assume that there exists a binary absorbing set $B \subsetneq A$ with the absorbing operation f. For any $(b_1, b_2) \in \omega$ and $b \in B$ we have $(f(b_1, b), f(b_2, b)) \in \omega \cap (B \times B)$. Then we can restrict σ and ω to B and apply the inductive assumption.

Thus, we assume that there does not exist an absorbing set $B \subsetneq A$ with the absorbing operation f. By Lemma 7.1, $\mathrm{pr}_1(\omega)$ binary absorbs A, $\mathrm{pr}_2(\omega)$ binary absorbs A. Then $\mathrm{pr}_1(\omega) = \mathrm{pr}_2(\omega) = A$. For every $b \in A$ we consider the set $A_b = \{a \mid (a, b) \in \sigma\}$ and $C_b = \{a \mid (a, b) \in \omega\}$. If $A_b = A$ then C_b is a binary absorbing set in A. Therefore $C_b = A$ and $(b, b) \in \omega$.

Assume that $A_b \neq A$ for some b. Since $b \in A_b$, we have $(A_b \times A_b) \cap \omega \neq \varnothing$. Then we restrict σ and ω to A_b and apply the inductive assumption. □

Lemma 7.3. *Suppose ρ is a binary absorbing set on $A_1 \times \dots \times A_n$. Then there exists a binary absorbing set B_i in A_i for some i.*

Proof. We prove by induction on the arity of ρ. If the projection of ρ onto the first coordinate is not A_1 then by Lemma 7.1 this projection is an absorbing set.

Otherwise, we choose any element $a \in A_1$ such that ρ does not contain all tuples with a as the first element.

Then we consider $\rho' = \{(a_2, \dots, a_n) \mid (a, a_2, \dots, a_n) \in \rho\}$, which is a binary absorbing subuniverse in $A_2 \times \dots \times A_n$. It remains to apply the induction assumption. □

A relation $\rho \subseteq A^n$ is called *C-essential* if $\rho \cap (C^{i-1} \times A \times C^{n-i}) \neq \varnothing$ for every i but $\rho \cap C^n = \varnothing$.

Lemma 7.4. *[1] Suppose C is a subuniverse of A. Then C absorbs A with an operation of arity n if and only if there does not exist a C-essential relation $\rho \subseteq A^n$.*

It is easy to check the following lemma.

Lemma 7.5. *Suppose $D^{(1)}$ is an absorbing reduction, the relation ρ is subdirect, then $\rho^{(1)}$ is not empty.*

7.2 Center

Lemma 7.6. *Suppose ρ is defined by a pp-formula $\Omega(x_1, \dots, x_n)$, Ω' is obtained from Ω by replacement of some constraint relations $\sigma_1, \dots, \sigma_s$ by constraint relations $\sigma'_1, \dots, \sigma'_s$ such that σ'_i is a center of σ_i for every i. Then the relation defined by $\Omega'(x_1, \dots, x_n)$ is a center of ρ.*

Proof. Suppose $\Omega'(x_1, \dots, x_n)$ defines the relation ρ'. Suppose $\mathbf{B}_i$ and R_i are the corresponding algebra and binary relation such that $\sigma'_i = \{c \mid \forall b \in B_i \colon (c, b) \in R_i\}$. Let $|B_i| = n_i$ for every i. Let Υ be obtained from Ω by replacement of every constraint $\sigma_i(y_1, \dots, y_t)$ by

$$R_i((y_1, \dots, y_t), z_{i,1}) \wedge \dots \wedge R_i((y_1, \dots, y_t), z_{i,n_i}).$$

Suppose $\Upsilon((x_1, \dots, x_n), (z_{1,1}, \dots, z_{s,n_s}))$ defines the relation R. It is not hard to see that $\rho' = \{c \mid \forall b \in (B_1^{n_1} \times \dots \times B_s^{n_s}) \colon (c, b) \in R\}$. By Lemma 7.3, there is no binary absorption on $B_1^{n_1} \times \dots \times B_s^{n_s}$. This proves that ρ' is a center of ρ. □

Corollary 7.6.1. , *Suppose* $\rho \subseteq A_1 \times \cdots \times A_n$ *is a relation such that* $\mathrm{pr}_1(\rho) = A_1$, $C = \mathrm{pr}_1((C_1 \times \cdots \times C_n) \cap \rho)$, *where* C_i *is a center in* A_i *for every* i. *Then* C *is a center in* A_1.

Corollary 7.6.2. *Suppose* C_i *is a center of* D_i *for every* i, *then* $C_1 \times \cdots \times C_n$ *is a center of* $D_1 \times \cdots \times D_n$.

Corollary 7.6.3. *Suppose* C_1 *and* C_2 *are centers of* D. *Then* $C_1 \cap C_2$ *is a center of* D.

In the proof of the following two lemmas we assume that a center C is defined by $C = \{a \in A \mid \forall b \in B\colon (a,b) \in R\}$ for a subalgebra R of $\mathbf{A} \times \mathbf{B}$. For an element $a \in A$ we put $a^+ = \{b \mid (a,b) \in R\}$.

Lemma 7.7. *Suppose* w *is a special WNU,* C *is a center of* A, *then* $w(c, c, \dots, c, a, c, \dots, c) \in C$ *for any* $a \in A$ *and* $c \in C$.

Proof. Assume the contrary. Put $w(c, \dots, c, a) = b \notin C$. Since w is a special WNU, we have $w(b, c, \dots, c) = b$ and $w(c, \dots, c, b) = b$. Then $w(b^+, B, \dots, B) \subseteq b^+$ and $w(B, \dots, B, b^+) \subseteq b^+$, and $w(x, \dots, x, y)$ defines a binary absorbing operation. This contradiction completes the proof. □

Lemma 7.8. *Suppose* w *is a special WNU of arity* m, C *is a proper center in* A, $\delta \subseteq A^s$ *is* C*-essential. Then* $s < (m \cdot |A|^{m^{|A|}})^{|A|}$.

Proof. Assume the contrary. For every i choose a tuple $\alpha_i \in \delta$ such that $\alpha_i \in C^{i-1} \times A \times C^{s-i}$. First, we introduce a quasi-order on elements of A. We say that $y_1 \leqslant y_2$ if $y_1^+ \subseteq y_2^+$, and $y_1 \sim y_2$ if $y_1^+ = y_2^+$. We can easily check that $b_1, b_2, \dots, b_n \geqslant c$ implies $w(b_1, \dots, b_n) \geqslant c$.

Suppose we have two tuples $(c_1, \dots, c_n)$, $(d_1, \dots, d_n)$, and $i \neq j$ such that $c_l \in C$ for every $l \neq i$, $d_l \in C$ for every $l \neq j$, $c_i \sim d_j$. It follows from the above argument that $w(c_1, \dots, c_n) \geqslant c_i$ and $w(d_1, \dots, d_n) \geqslant d_j$. If $c_i \sim w(c_1, \dots, c_n)$ and $d_j \sim w(d_1, \dots, d_n)$ then $w(\underbrace{B, \dots, B}_{i-1}, c_i^+, B, \dots, B) \subseteq c_i^+$ and $w(\underbrace{B, \dots, B}_{j-1}, c_i^+, B, \dots, B) \subseteq c_i^+$. Therefore, the formula $w(\underbrace{x, \dots, x}_{i-1}, y, x, \dots, x)$ defines a binary absorbing operation on B, which contradicts the definition of a center.

We say that an element is *foreign* if it is not from the center. We say that tuples are *independent* if they do not have foreign elements on the same position. We start with s tuples $\alpha_1, \dots, \alpha_s$. On every step we exclude at least one element of A from all tuples.

Assume that we have independent tuples $\beta_1, \dots, \beta_{s_i}$. Choose a minimal element appearing in βs. Let it be c. Assume that the foreign elements of β_1 appear in the positions $j_1, \dots, j_h$. Then we choose the most popular projection of tuples $\beta_2, \dots, \beta_{s_i}$ onto coordinates $j_1, \dots, j_h$, and remove all tuples but β_1 with a different projection. Our independent set became smaller. Without loss of generality we assume that $\beta_1, \dots, \beta_{s_i'}$ is the obtained independent set.

We know that there can be only one position of d such that $w(\dots, d, \dots) \sim c$ for some $d \sim c$. Without loss of generality we assume that this is the first position. Then we generate new independent tuples in the following way $\beta_1' = w(\beta_1, \beta_2, \dots, \beta_m)$, $\beta_2' = w(\beta_1, \beta_{m+1}, \beta_{m+2}, \dots, \beta_{2m-1})$ and so on. It remains to show that there are no elements equivalent to c in the obtained tuples.

By Lemma 7.7, the j_k-th element of every new tuple is a central element. We cannot get such an element in the remaining positions because $w(\dots, d, \dots) > c$ for every $d \sim c$ (we put d not in the first position).

Thus, we exclude at least one element on every step. Hence in $|A|$ steps we get a tuple where all elements are from the center. Therefore, on every step we have less than $m^{|A|}$ foreign elements in each tuple. Hence, on every step we decrease the number of tuples by a factor of at most $|A|^{m^{|A|}}$ (because we choose tuples that are the same in some coordinates) and by m (because from m tuples we obtain just one). Thus, if the original number of tuples is at least $(m \cdot |A|^{m^{|A|}})^{|A|}$ then we get a tuple where all elements are from the center. This contradicts $C^s \cap \delta = \varnothing$. □

Combining this result with Lemma 7.4, we obtain the following corollary.

Corollary 7.8.1. *Suppose C is a center of A, then C is a absorbing subuniverse of A.*

The following lemma is a stronger version of the original lemma suggested by Marcin Kozik.

Lemma 7.9. *Suppose $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ are centers, B is a subuniverse of D, a relation $\rho \subseteq A_1 \times D^l \times A_2$ satisfies the following properties: $(C_1 \times B^l \times C_2) \cap \rho = \varnothing$, for every $i \in \{1, 2, \dots, l\}$*

$$(C_1 \times B^{i-1} \times D \times B^{l-i} \times C_2) \cap \rho \neq \varnothing,$$

$$(A_1 \times B^l \times C_2) \cap \rho \neq \varnothing, \quad (C_1 \times B^l \times A_2) \cap \rho \neq \varnothing.$$

Then there exists a relation $\rho' \subseteq A_1 \times D^{2l} \times A_1$ such that for every $i \in \{1, 2, \dots, 2l\}$

$$(C_1 \times B^{i-1} \times D \times B^{2l-i} \times C_1) \cap \rho' \neq \varnothing,$$

$$(A_1 \times B^{2l} \times C_1) \cap \rho' \neq \varnothing, \quad (C_1 \times B^{2l} \times A_1) \cap \rho \neq \varnothing, (C_1 \times B^{2l} \times C_1) \cap \rho' = \varnothing.$$

Proof. Assume that ρ is a minimal by inclusion relation of arity $l+2$ satisfying the above properties. Put $E = \mathrm{pr}_{l+2}(\rho \cap (C_1 \times B^l \times A_2))$. Since ρ is minimal, for any $b \in E$ the algebra generated by $\{b\} \cup C_2$ contains $\mathrm{pr}_{l+2}(\rho)$. Fix $b \in E$.

Let σ be the subalgebra of $A_2 \times A_2$ generated by $\{b\} \times C_2 \cup C_2 \times C_2 \cup C_2 \times \{b\}$. Put

$$\rho'(x, y_1, \dots, y_l, y'_1, \dots, y'_l, x') = \exists z \exists z' \; \rho(x, y_1, \dots, y_l, z) \wedge \rho(x', y'_1, \dots, y'_l, z') \wedge \sigma(z, z').$$

It is not hard to see that ρ' satisfies all necessary conditions, possibly except for the last one. Assume that $(C_1 \times B^{2l} \times C_1) \cap \rho' \neq \varnothing$ and the tuple in the intersection is obtained by sending z to d and z' to d'. Clearly, $d, d' \in E$ and $\{e \in A_2 \mid (e, d') \in \sigma\} \supseteq \{d\} \cup C_2$, therefore $\{e \in A_2 \mid (e, d') \in \sigma\} \supseteq \mathrm{pr}_{l+2}(\rho)$. Hence, $\{e \in A_2 \mid (b, e) \in \sigma\} \supseteq \{d'\} \cup C_2$ and $\{e \in A_2 \mid (b, e) \in \sigma\} \supseteq \mathrm{pr}_{l+2}(\rho)$.

Thus, $(b, b) \in \sigma$ and there exists an n-ary term t such that

$$t(b, b, \dots, b, c_1, \dots, c_i) = b, \quad t(c'_1, \dots, c'_j, b, b, \dots, b) = b,$$

where $i + j \geqslant n$ and $c_1, \dots, c_i, c'_1, \dots, c'_j \in C_2$. Suppose $R \subseteq A_2 \times G$ is a binary relation from the definition of the center C_2, $b^+ = \{a \mid (b, a) \in R\}$. Then, b^+ absorbs G with the binary term $t(\underbrace{x, \dots, x}_{j}, y, \dots, y)$. This contradiction completes the proof. □

Corollary 7.9.1. *Suppose $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ are centers, $B \subseteq D$ is an absorbing subuniverse, $\rho \subseteq A_1 \times D \times A_2$ is a ternary relation such that $(C_1 \times D \times C_2) \cap \rho \neq \varnothing$, $(C_1 \times B \times A_2) \cap \rho \neq \varnothing$, $(A_1 \times B \times C_2) \cap \rho \neq \varnothing$. Then $(C_1 \times B \times C_2) \cap \rho \neq \varnothing$.*

Proof. Assume the contrary. By Lemma 7.9 we can increase the arity of ρ as much as we need. If we restrict the first and the last variables to the corresponding centers and consider the projection onto the remaining variables we get a C-essential relation. This contradicts the fact that C is an absorbing subuniverse. □

Corollary 7.9.2. *Suppose C is a center of A, then C is a ternary absorbing subuniverse of A.*

Corollary 7.9.3. *Suppose $\rho \subseteq A_1 \times \cdots \times A_k$ is a relation, $k \geqslant 3$, C_i is a center in A_i and $\rho \cap (C_1 \times \cdots \times C_{i-1} \times A_i \times C_{i+1} \times \cdots \times C_k) \neq \varnothing$ for every i. Then $\rho \cap (C_1 \times \cdots \times C_k) \neq \varnothing$.*

7.3 PC Subuniverse

Lemma 7.10. *Suppose $\rho \subseteq A \times B$ is a subdirect relation, A is a PC algebra. Then either for every $c \in B$ there exists a unique $a \in A$ such that $(a, c) \in \rho$, or there exists $c \in B$ such that $(a, c) \in \rho$ for every $a \in A$.*

Proof. Put $\sigma_l(x_1, x_2, \dots, x_l) = \exists y\ \rho(x_1, y) \wedge \dots \wedge \rho(x_l, y)$. Since A is PC algebra, σ_2 is either full, or the equality relation.

If σ_2 is the equality relation, then for every $c \in B$ there exists a unique $a \in A$ such that $(a, c) \in \rho$.

Suppose σ_2 is full. Then we consider the minimal l such that σ_l is not full. It is easy to see that σ_l cannot be preserved by all operations on A. This contradiction proves that $\sigma_{|A|}$ is also full. This means that for some c we have $(a, c) \in \rho$ for every $a \in A$. □

Lemma 7.11. *Suppose $\rho \subseteq A_1 \times \dots \times A_n$ is a subdirect relation, A_i is a PC algebra for every $i \in \{2, \dots, n\}$, there is no binary absorption and center on A_i for every $i \in \{1, \dots, n\}$. Then ρ can be represented as a conjunction of binary relations $\delta_1, \dots, \delta_k$ such that* $\mathrm{Con}(\delta_l, j)$ *is the equality relation whenever the domain of the j-th variable of δ_l is a PC algebra.*

Proof. Assume the contrary. Let us consider a relation of the minimal arity such that the lemma does not hold.

Assume that ρ is not essential, then it can be represented as a conjunction of essential relations satisfying the same properties. By the inductive assumption, each of them can be represented as a conjunction of binary relations. It remains to join these binary relation to complete the proof for this case.

Assume that ρ is essential. The projection of ρ onto any set of variables gives a relation of a smaller arity satisfying the same properties. By the inductive assumption, the relation of a smaller arity can be represented as a conjunction of binary relations $\delta_1, \dots, \delta_k$ such that $\mathrm{Con}(\delta_l, j)$ is the equality relation whenever the domain of the j-th variable of δ_l is a PC algebra. Since ρ is essential, the relation of smaller arity is a full relation.

Let us consider the relation $\rho \subseteq (A_1 \times \dots \times A_{n-1}) \times A_n$ as a binary relation. By Lemma 7.10 we have one of the following two situations.

Case 1: there exist $b_1, \dots, b_{n-1}$ such that $(b_1, \dots, b_{n-1}, a) \in \rho$ for every $a \in A_n$. We consider the maximal s such that $\rho(b_1, \dots, b_s, x_{s+1}, \dots, x_n)$ is not a full relation. Obviously $s \leqslant n-2$ and s exists. Then we get a proper center C on A_{s+1} defined by $C = \{a_{s+1} \in A_{s+1} \mid \forall a_{s+2} \dots \forall a_n \colon (b_1, \dots, b_s, a_{s+1}, a_{s+2}, \dots, a_n) \in \rho\}$.

Case 2: for every $a_1, \dots, a_{n-1}$ there exists a unique b such that $(a_1, \dots, a_{n-1}, b) \in \rho$. In the same way we can show that for any $i \in \{2, \dots, n\}$ and $(a_1, \dots, a_n)$ there exists a unique b such that $(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) \in \rho$. Thus, if ρ is binary, then the statement is proved.

If the arity of ρ is greater than 2, then the following formula defines a subdirect relation on A_n such that the projection onto any 3 coordinates is a full relation:

$$\zeta(z_1, z_2, z_3, z_4) = \exists x_1 \exists x_2 \dots \exists x_{n-1} \exists x_1' \exists x_2'\ \rho(x_1, x_2, x_3, \dots, x_{n-1}, z_1) \wedge \rho(x_1, x_2', x_3, \dots, x_{n-1}, z_2) \wedge \rho(x_1', x_2, x_3, \dots, x_{n-1}, z_3) \wedge \rho(x_1', x_2', x_3, \dots, x_{n-1}, z_4).$$

We can check that if $(a_1, a_2, a_3, a_4) \in \zeta$ then $(a_1 = a_2) \Leftrightarrow (a_3 = a_4)$. This contradicts the fact that A_n is a PC algebra. □

It follows from Lemma 7.11 that the quotient of any algebra $\mathbf{A}$ without a center and binary absorption by $\mathrm{ConPC}(\mathbf{A})$ is a direct product of PC algebras.

Corollary 7.11.1. *Suppose $\rho \subseteq A_1 \times \dots \times A_n$ is a subdirect relation, there is no binary absorption and center on A_i for every i, $C = \mathrm{pr}_1((C_1 \times \dots \times C_n) \cap \rho)$, where C_i is a PC subuniverse in A_i for every i. Then C is a PC subuniverse in A_1.*

Corollary 7.11.2. *Suppose $\rho \subseteq A_1 \times \cdots \times A_k$ is a subdirect relation, $k \geqslant 3$, C_i is a PC subuniverse in A_i, there is no binary absorption and center on A_i for every i, $\rho \cap (C_1 \times \cdots \times C_{i-1} \times A_i \times C_{i+1} \times \cdots \times C_k) \neq \varnothing$ for every i. Then $\rho \cap (C_1 \times \cdots \times C_k) \neq \varnothing$.*

Lemma 7.12. *Suppose $\rho \subseteq A \times B$ is a subdirect relation, A is a PC algebra without center and binary absorption, $C = \{b \in B \mid \forall a \in A\colon (a,b) \in \rho\}$. Then C binary absorbs B.*

Proof. Suppose $A = \{a_1, \ldots, a_k\}$. Let us consider the matrix M whose rows are tuples $(\underbrace{a, a, \ldots, a}_{k+1}, b, a_1, \ldots, a_k)$ and $(b, a_1, \ldots, a_k, \underbrace{a, a, \ldots, a}_{k+1})$ for all $a, b \in A$. The $2k+2$ columns of this matrix we denote by $\alpha_1, \ldots, \alpha_{2k+2}$. By β we denote the tuple of length $2k^2$ such that the i-th element of β equals b from the corresponding row. By Lemma 7.11, the relation generated by $\alpha_1, \ldots, \alpha_{2k+2}$ is a full relation. Hence, there exists a term operation f such that $f(\alpha_1, \ldots, \alpha_{2k+2}) = \beta$. Let us show that C absorbs B with the term operation defined by $h(x, y) = f(\underbrace{x, \ldots, x}_{k+1}, y, \ldots, y)$. Suppose $d \in B$, $c \in C$. Assume that $h(d, c) = e \notin C$. Choose elements $a, a' \in A$ such that $(a, e) \notin \rho$ and $(a', d) \in \rho$. Consider the row $(a', \ldots, a', a, a_1, \ldots, a_k)$ from the matrix. We know that f returns a on this tuple and $f(\underbrace{d, \ldots, d}_{k+1}, c, \ldots, c) = e$, which contradicts the fact that f preserves ρ.

In the same way we prove that $h(c, d) \in C$ for every $d \in B$, $c \in C$. □

Lemma 7.13. *Suppose $\rho \subseteq A \times B \times B$ is a subdirect relation, A is a PC algebra without a center and a binary absorption, for every $b \in B$ there exists $a \in A$ such that $(a, b, b) \in \rho$. Then for every $a \in A$ there exists $b \in B$ such that $(a, b, b) \in \rho$.*

Proof. We prove by induction on the size of B.

By Lemma 7.10, only two situations are possible: either there exists $c_1, c_2 \in B$ such that $(a, c_1, c_2) \in \rho$ for every $a \in A$, or for all $b_1, b_2 \in B$ there exists a unique $a \in A$ such that $(a, b_1, b_2) \in \rho$. Consider the first case. Put $D = \{(b, c) \mid \forall a \in A\colon (a, b, c) \in \rho\}$. By Lemma 7.12, D is a binary absorbing subuniverse in the projection of ρ onto the last two variables. By Lemma 7.2, there exists $(b, b) \in D$. This completes this case.

Consider the second case. Let δ_1 be the projection of ρ onto the first two variables. By Lemma 7.10 we have one of two situations. Assume that for every $b \in B$ there exists a unique a such that $(a, b) \in \delta_1$. Then we can check that if $(a, b, b') \in \rho$ then $(a, b, b) \in \rho$, which completes this case. Otherwise, there exists an element b such that $(a, b) \in \delta_1$ for every $a \in A$. Consider the relation $\delta_2(x, y_2) = \rho(x, b, y_2)$. If $\mathrm{pr}_2(\delta_2) \neq B$, then we restrict the last two variables of ρ to $\mathrm{pr}_2(\delta_2)$ and apply the inductive assumption. Assume that $\mathrm{pr}_2(\delta_2) = B$. By the definition of the second case we know that for every $c \in B$ there exists a unique a such that $(a, c) \in \delta_2$. Then there exists a congruence σ on B such that B/σ is a PC algebra. If σ is the equality relation, then B is a PC algebra without center and binary absorption. Then the statement follows from Lemma 7.11.

If σ is not the equality relation, then we consider the relation ρ' obtained from ρ by factorization of the last two variables by σ. By the inductive assumption for any $a \in A$ there exists $E \in B/\sigma$ such that $(a, E, E) \in \rho'$. By Lemma 7.10, we have one of the following situations. Case 1. There exists $E \in B/\sigma$ such that for every $a \in A$ we have $(a, E, E) \in \rho'$. Then we restrict the last two variables of ρ to E and apply the inductive assumption. Case 2. For every $E \in B/\sigma$ there exists a unique $a \in A$ such that $(a, E, E) \in \rho'$. In this case for any $a \in A$ we choose E such that $(a, E, E) \in \rho'$. By the above condition we have $(a, b, b) \in \rho$ for any $b \in E$, which completes the proof. □

7.4 Linear Subuniverse

Lemma 7.14. *Suppose $\rho \subseteq A_1 \times A_2$ is a subdirect relation, A_2 is a linear algebra, no binary absorption on A_1. Then for all $a, b \in A_1$ we have*

$$|\{c \mid (a,c) \in \rho\}| = |\{c \mid (b,c) \in \rho\}|.$$

Proof. Assume the contrary, then we choose all elements a with the maximal $|\{c \mid (a,c) \in \rho\}|$. Denote the set of such elements by C.

Since $w(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_m)$ is a bijection on A_2 for every $a_1, \ldots, a_m \in A_2$, we have $w(A_1, \ldots, A_1, C, A_1, \ldots, A_1) \subseteq C$. Hence $w(x, \ldots, x, y)$ is a binary absorbing operation and C is a binary absorbing set. □

Lemma 7.15. *Suppose $\rho \subseteq A_1 \times A_2$ is a subdirect relation, A_2 is a linear algebra, no binary absorption on A_1. Then ρ has the parallelogram property.*

Proof. First, we define a relation σ_k for every $k \geqslant 2$ by

$$\sigma_k(y_1, \ldots, y_k) = \exists x\ \rho(x, y_1) \wedge \cdots \wedge \rho(x, y_k).$$

Since σ_k is preserved by the Mal'tsev operation $w(x, y, \ldots, y, z)$ and reflexive, σ_2 is a congruence. Let us show by induction on k that $\sigma_k(y_1, \ldots, y_k) = \bigwedge_{i=2}^{k} \sigma_2(y_1, y_i)$. Let k be the minimal number such that $(a_1, \ldots, a_k) \notin \sigma_k$ and $(a_i, a_j) \in \sigma_2$ for every i, j. Then $(a_1, a_1, a_3, \ldots, a_k), (a_1, a_2, a_1, a_4, \ldots, a_k) \in \sigma_k$. Therefore $(a_1, a_1, a_1, a_4 \ldots, a_k) \notin \sigma_k$, which contradicts our assumption.

Thus, for every equivalence class E of σ_2 there exists $c \in A_1$ such that $(c, d) \in \rho$ for any $d \in E$. It follows from Lemma 7.14, that ρ has parallelogram property. □

Corollary 7.15.1. *Suppose $\rho \subseteq A_1 \times \cdots \times A_n$ is a relation such that $\mathrm{pr}_1(\rho) = A_1$, there is no binary absorption on A_1, $C = \mathrm{pr}_1((C_1 \times \cdots \times C_n) \cap \rho)$, where C_i is a linear subuniverse in A_i for every i. Then C is a linear subuniverse in A_1.*

8 Proof of the Auxiliary Statements

8.1 Reductions preserve cycle-consistency and irreducibility

Lemma 8.1. *Suppose $D^{(1)}$ is a proper minimal reduction, the constraint $\rho(x_1, \ldots, x_n)$ is subdirect, $\rho^{(1)}$ is not empty. Then $\rho^{(1)}$ is subdirect.*

Proof. By Corollaries 7.1.1, 7.6.1, 7.11.1, 7.15.1, if we restrict the variables $x_1, x_2, \ldots, x_n$ of ρ to $D^{(1)}$, then we restrict the projection correspondingly. Since $D^{(1)}_{x_i}$ is a minimal absorbing subuniverse, a minimal center, a minimal PC subuniverse, or a minimal linear subuniverse, the relation $\rho^{(1)}(x_1, \ldots, x_n)$ is subdirect. □

Lemma 8.2. *Suppose $D^{(1)}$ is a proper minimal reduction for a cycle-consistent irreducible CSP instance Θ, $\Theta^{(1)}$ has a solution. Then $\Theta^{(1)}$ is cycle-consistent and irreducible.*

Proof. Consider a path $z_1 - C_1 - z_2 - \cdots - C_{l-1} - z_l$ starting and ending with one variable x. By Ω we denote the formula corresponding to this path, that is a formula obtained from the path such that every variable except for $z_2, \ldots, z_{l-1}$ occurs just once, $z_2, \ldots, z_{l-1}$ occur twice. Let $\{z_1, \ldots, z_l, y_1, \ldots, y_t\}$ be the set of all variables appearing in Ω. By Ω' we denote the formula obtained from Ω by replacement of z_l by z_1.

First, we want to prove that this path connects a with a in $\Theta^{(1)}$ for every $a \in D_x^{(1)}$. Second, we prove that if the path connects any two elements of D_x, then it connects any two elements of $D_x^{(1)}$.

Assume that $D^{(1)}$ is not a PC reduction. We know that $\Omega'(z_1)$ defines D_x. Since $\Theta^{(1)}$ has a solution, Corollaries 7.1.1, 7.6.1, 7.15.1 imply that $\Omega'^{(1)}(z_1)$ defines the corresponding subuniverse in D_x. Since $D^{(1)}$ is minimal, this subuniverse is equal to $D_x^{(1)}$. Hence, this path connects a with a in $\Theta^{(1)}$ for every $a \in D_x^{(1)}$. Assume that the path connects any two elements of D_x. Then $\Omega(z_1, z_l)$ contains all pairs $(a, a') \in D_x \times D_x$. Combining Corollaries 7.1.1, 7.6.1, 7.15.1 with the fact that $D^{(1)}$ is a minimal reduction, we prove that $\Omega^{(1)}(z_1, z_l)$ contains all pairs $(a, a') \in D_x^{(1)} \times D_x^{(1)}$.

Suppose $D^{(1)}$ is a PC reduction. Let $\Omega(z_1, \dots, z_{l-1}, z_l, y_1, \dots, y_t)$ define a relation ρ. We factorize variables $z_2, \dots, z_{l-1}, y_1, \dots, y_t$ of ρ by (1) and replace by PC variables. As a result we get a relation $\delta(z_1, z_l, u_1, \dots, u_k)$, where $u_1, \dots, u_k$ are PC variables. By Lemma 7.13, if we identify z_1 and z_l in δ then we do not restrict any variable u_i. Therefore $\delta(z_1, z_1, u_1, \dots, u_k)$ defines a subdirect relation. By Corollary 7.11.1, if we restrict variables $u_1, \dots, u_k$ of this subdirect relation to $D^{(1)}$, then we restrict the variable z_1 to a PC subuniverse. Since $D^{(1)}$ is minimal and $\Theta^{(1)}$ has a solution, the path connects a with a in $\Theta^{(1)}$ for every a. Thus, we proved that $\Theta^{(1)}$ is cycle-consistent.

Assume that the path connects any two elements of D_x. By Lemma 7.11, if we put $z_1 = a$ in δ then we restrict a variable u_i to one-element set or do not restrict at all. Therefore, by Corollary 7.11.1, if we put $z_1 = a$ and restrict $u_1, \dots, u_k$ to $D^{(1)}$, then we restrict z_l to a PC subuniverse. Since $D^{(1)}$ is minimal, the path connects any two elements of $D_x^{(1)}$.

Let us prove that $\Theta^{(1)}$ is irreducible. Assume the contrary. Consider a formula Υ_1 consisting of projections of constraints from $\Theta^{(1)}$ such that it is not fragmented, not linked and its solution set is not subdirect. Let $\mathrm{Var}(\Upsilon_1) = \{x_1, \dots, x_n\}$. It is not hard to find an instance $\Upsilon \in \mathrm{Coverings}(\Theta)$ such that $\Upsilon_1(x_1, \dots, x_n) = \Upsilon^{(1)}(x_1, \dots, x_n)$ (every variable except for $x_1, \dots, x_n$ appears just once). By Lemma 6.1, Υ is irreducible.

Assume that Υ is linked. Consider a path that connects any two elements of D_{x_1} in Υ. By the above argument, it also connects any two elements of $D_{x_1}^{(1)}$ in $\Upsilon^{(1)}$. Therefore, Υ_1 is also linked, which contradicts our assumption.

Suppose Υ is not linked. Since Υ is irreducible, the solution set of Υ is subdirect. Let $v_1, \dots, v_r$ be the remaining variables of Υ. By Corollaries 7.1.1, 7.6.1, 7.15.1, 7.11.1 the restriction of $v_1, \dots, v_r$ and $x_1, \dots, x_n$ to $D^{(1)}$ implies the corresponding restrictions of $x_1, \dots, x_n$. Since the reduction $D^{(1)}$ is minimal, the relation defined by $\Upsilon^{(1)}(x_1, \dots, x_n)$ is subdirect. □

8.2 Properties of $\mathrm{Con}(\rho, x)$

Lemma 8.3. *Suppose ρ is a critical subdirect relation, the i-th variable of ρ is rectangular. Then $\mathrm{Con}(\rho, i)$ is an irreducible congruence.*

Proof. To simplify notations assume that $i = 1$. Put $\sigma = \mathrm{Con}(\rho, i)$. Assume the contrary. Consider binary relations $\delta_1, \dots, \delta_s$ compatible with σ such that $\delta_1 \cap \dots \cap \delta_s = \sigma$. Put

$$\rho_i(x_1, \dots, x_n) = \exists x_1' \; \rho(x_1', x_2, \dots, x_n) \wedge \delta_i(x_1, x_1').$$

It is easy to see that the intersection of $\rho_1, \dots, \rho_s$ gives ρ, which contradicts the fact that ρ is critical. □

Lemma 8.4. *Suppose σ, σ_1, and σ_2 are congruences on A, $\sigma \cap \sigma_1 = \sigma \cap \sigma_2$, $\sigma \setminus \sigma_1 \neq \varnothing$. Then σ_1 and σ_2 are adjacent.*

Proof. Let us define a relation ρ by

$$\rho(x_1, x_2, y_1, y_2) = \exists z_1 \exists z_2\ \sigma_1(x_1, z_1) \wedge \sigma_2(z_1, y_1) \wedge \sigma_1(x_2, z_2) \wedge \sigma_2(z_2, y_2) \wedge \sigma(z_1, z_2).$$

It is easy to check for any $(a_1, a_2, a_3, a_4) \in \rho$ that $(a_1, a_2) \in \sigma_1 \Leftrightarrow (a_3, a_4) \in \sigma_2$. Also $(a, a, a, a) \in \rho$ for any $a \in A$. Choose $(a, b) \in \sigma \setminus \sigma_1$. Then $(a, b, a, b) \in \rho$, which proves that ρ is a reflexive bridge. □

For a relation ρ of arity n by $\mathrm{UnPol}(\rho)$ we denote the set of all unary vector-functions preserving the relation ρ.

Suppose Σ is a set of constraints with the same scope of variables. For a tuple α we say that a constraint C is *maximal without α in Σ* if there does not exist a weaker constraint $C' \in \Sigma$ such that α does not satisfy C'.

Lemma 8.5. *Suppose a pp-formula $\Omega(x_1, \ldots, x_n)$ defines a relation ρ, $\alpha \notin \rho$, $\rho' = \{f(\alpha) \mid f \in \mathrm{UnPol}(\rho)\}$. Then there exists $\Omega' \in \mathrm{Coverings}(\Omega)$ such that $\Omega'(x_1, \ldots, x_n)$ defines ρ'.*

Proof. Suppose $\alpha = (a_1, \ldots, a_n)$. We introduce new variables x_i^a for every $i \in \{1, 2, \ldots, n\}$ and $a \in D_{x_i}$. By Υ we denote the following formula $\bigwedge\limits_{(b_1,\ldots,b_n)\in\rho} \rho(x_1^{b_1}, \ldots, x_n^{b_n})$. It is easy to see that ρ' is defined by a pp-formula $\Upsilon(x_1^{a_1}, \ldots, x_n^{a_n})$. To obtain formula Ω' it is sufficient to replace every occurrence of ρ by Ω with the corresponding variables. □

Corollary 8.5.1. *Suppose Ω is a formula, Σ is the set of all constraints defined by $\Upsilon(x_1, \ldots, x_n)$ where $\Upsilon \in \mathrm{Coverings}(\Omega)$, C is a maximal constraint in Σ without a tuple α. Then α is a key tuple for the constraint relation of C.*

Proof. Suppose $C = \rho(x_1, \ldots, x_n)$. For every tuple $\beta \notin \rho$ we consider $\rho_\beta := \{f(\beta) \mid f \in \mathrm{UnPol}(\rho)\}$. It is easy to see that $\rho_\beta \supsetneq \rho$ for every β. By Lemma 8.5, ρ_β can be defined by a constraint from Σ. Since C is maximal, $\alpha \in \rho_\beta$. Therefore, α is a key tuple for ρ. □

Lemma 8.6. *Suppose $D^{(1)}$ is a minimal nonlinear reduction for a formula Υ, the solution set of Υ is subdirect, $\Upsilon^{(1)}(x_1, \ldots, x_n)$ defines a subdirect rectangular relation. Then for every i*

$$(\mathrm{Con}(\Upsilon(x_1, \ldots, x_n), x_i))^{(1)} = \mathrm{Con}(\Upsilon^{(1)}(x_1, \ldots, x_n), x_i).$$

Proof. WLOG we prove for $i = 1$. Let $\{x_1, \ldots, x_n, y_1, \ldots, y_s\}$ be the set of all variables of Υ. Define the relation ρ by $\Upsilon(x_1, \ldots, x_n, y_1, \ldots, y_s)$. Put $\sigma_0 = \mathrm{Con}(\Upsilon(x_1, \ldots, x_n), x_1)$, $\sigma_1 = \mathrm{Con}(\Upsilon^{(1)}(x_1, \ldots, x_n), x_1)$,

$$\rho'(x_1, x_2, \ldots, x_n, y_1, \ldots, y_s, y_1', \ldots, y_s', x_1') = \rho(x_1, x_2, \ldots, x_n, y_1, \ldots, y_s) \wedge \rho(x_1', x_2, \ldots, x_n, y_1', \ldots, y_s').$$

Assume the opposite. Choose a pair $(a, b) \in \sigma_0^{(1)} \setminus \sigma_1$. Then there exists α such that $a\alpha b \in \rho'$.

Since $\Upsilon^{(1)}(x_1, \ldots, x_n)$ defines a subdirect relation, for every $c \in D_{x_1}^{(1)}$ there exists β_c such that $c\beta_c c \in \rho'^{(1)}$. Consider tuples $\xi_0 = a\alpha b\beta_b b\beta_b b$, $\xi_1 = a\beta_a a\alpha b\beta_b b$, $\xi_2 = a\beta_a a\beta_a a\alpha b$. If $D^{(1)}$ is a central reduction, then by Lemma 7.9.2, there exists a ternary absorbing term operation t. Then $t(\xi_0, \xi_1, \xi_2)$ defines a path from a to b with edges from σ_1. If $D^{(1)}$ is an absorbing reduction, then $t'(\xi_0, \xi_1)$ defines a path from a to b with edges from σ_1, where t' is the binary absorbing operation. Since $\Upsilon^{(1)}(x_1, \ldots, x_n)$ defines a rectangular relation, σ_1 is a congruence. Therefore, there is no path from a to b in σ_1.

Assume that $D^{(1)}$ is a PC reduction. We factorize variables $x_2, \ldots, x_n, y_1, \ldots, y_s, y_1', \ldots, y_s'$ of ρ' by (1), replace every new variable by a set of PC variables, and restrict variables x_1 and

$x_{1'}$ to $D_{x_1}^{(1)}$. As a result we obtain the relation $\rho''(x_1, z_1, \dots, z_k, x_1')$, where the domain of z_i is a PC algebra for every i. By Corollary 7.11.1, every variable z_i in ρ'' either takes on all values from the domain or just one value. By Lemma 8.1, $\rho^{(1)}$ is subdirect. Then, without loss of generality we assume that the relation ρ'' is subdirect (otherwise we consider the projection of ρ'' onto all variables taking on more than 1 value).

By 0 we denote the element of every PC algebra corresponding to the reduction $D^{(1)}$. Since $\rho^{(1)}$ is subdirect, $(c, 0, \dots, 0, c) \in \rho''$ for every $c \in D_{x_1}^{(1)}$. Lemma 7.11 implies that for every $c \in D_{x_1}^{(1)}$ the formula $\exists x_1' \rho''(c, z_1, \dots, z_k, x_1')$ defines a subdirect relation. We also know that there exist $c_1, \dots, c_k$ such that $(a, c_1, \dots, c_k, b) \in \rho''$. Put

$$\rho_0(z_1, \dots, z_{4k}) = \exists x' \exists y \exists y' \; \rho''(a, z_1, \dots, z_k, y) \wedge \rho''(x', z_{k+1}, \dots, z_{2k}, y) \wedge \\ \rho''(x', z_{2k+1}, \dots, z_{3k}, y') \wedge \rho''(b, z_{3k+1}, \dots, z_{4k}, y').$$

Since we can put $x' = a$ or $x' = b$, ρ_0 is subdirect. We can check that

$$(c_1, \dots, c_k, 0, \dots, 0, 0, \dots, 0, 0, \dots, 0) \in \rho_0 \text{ (for } x' = y = y' = b);$$
$$(0, \dots, 0, 0, \dots, 0, c_1, \dots, c_k, 0, \dots, 0) \in \rho_0 \text{ (for } x' = y = a,\ y' = b);$$
$$(c_1, \dots, c_k, c_1, \dots, c_k, c_1, \dots, c_k, 0, \dots, 0) \in \rho_0 \text{ (for } x' = a,\ y = y' = b)$$

but $(0, \dots, 0) \notin \rho_0$. By Lemma 7.11, ρ_0 can be represented as a conjunction of bijective binary relations, and none of them can omit the tuple $(0, \dots, 0, 0, \dots, 0, 0, \dots, 0, 0, \dots, 0)$. Contradiction. □

Lemma 8.7. *Suppose $D^{(1)}$ is a minimal linear reduction for Υ, $\Upsilon^{(1)}(x_1, \dots, x_n)$ defines a subdirect rectangular relation,* $\mathrm{Var}(\Upsilon) = \{x_1, \dots, x_n, v_1, \dots, v_r\}$, $\Omega = \Upsilon \wedge \bigwedge_{i=1}^{r} \sigma_i(v_i, u_i)$, *where* $\sigma_i = \mathrm{ConLin}(D_{v_i})$. *Then* $(\mathrm{Con}(\Omega(x_1, \dots, x_n, u_1, \dots, u_r), x_j))^{(1)} = \mathrm{Con}(\Upsilon^{(1)}(x_1, \dots, x_n), x_j))$ *for every j.*

Proof. Without loss of generality assume that $j = 1$. Suppose $\Omega(x_1, \dots, x_n, u_1, \dots, u_r)$ and $\Upsilon^{(1)}(x_1, \dots, x_n)$ define the relations ρ' and ρ correspondingly. Assume the opposite. Then there exist $a, b \in D_{x_1}^{(1)}$ such that $(a, b) \in \mathrm{Con}(\rho', 1) \setminus \mathrm{Con}(\rho, 1)$. Therefore for some β we have $a\beta, b\beta \in \rho'$. Since ρ is subdirect, there exist α_a and α_b in $D^{(1)}$ such that $a\alpha_a, b\alpha_b \in \rho'$. It is easy to check that

$$w(a, a, \dots, a) w(\alpha_a, \beta, \dots, \beta) \in \rho',$$
$$w(a, b, \dots, b) w(\alpha_a, \beta, \dots, \beta) \in \rho',$$
$$w(a, b, \dots, b) w(\alpha_b, \beta, \dots, \beta) \in \rho',$$
$$w(b, b, \dots, b) w(\alpha_b, \beta, \dots, \beta) \in \rho'.$$

Since w is a special WNU, $w(\alpha_a, \beta, \dots, \beta)$ and $w(\alpha_b, \beta, \dots, \beta)$ belong to $D^{(1)}$, for $c = w(a, b, \dots, b)$ we have $(a, c), (c, b) \in \mathrm{Con}(\rho, 1)$. Since ρ is rectangular, we have $(a, b) \in \mathrm{Con}(\rho, 1)$. This contradiction completes the proof. □

8.3 Adding linear variable

Below we formulate few statements from [21] that will help us to prove the main property of a bridge. A relation $\rho \subseteq A^n$ is called *strongly rich* if for every tuple $(a_1, \dots, a_n)$ and every $j \in \{1, \dots, n\}$ there exists a unique $b \in A$ such that $(a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n) \in \rho$. We will need two statements from [21].

Theorem 8.8. *[21] Suppose $\rho \subseteq A^n$ is a strongly rich relation preserved by an idempotent WNU. Then there exists an abelian group $(A;+)$ and bijective mappings $\phi_1, \phi_2, \ldots, \phi_n : A \to A$ such that*

$$\rho = \{(x_1, \ldots, x_n) \mid \phi_1(x_1) + \phi_2(x_2) + \ldots + \phi_n(x_n) = 0\}.$$

Lemma 8.9. *[21] Suppose $(G;+)$ is a finite abelian group, the relation $\sigma \subseteq G^4$ is defined by $\sigma = \{(a_1, a_2, a_3, a_4) \mid a_1 + a_2 = a_3 + a_4\}$, σ is preserved by an idempotent WNU f. Then $f(x_1, \ldots, x_n) = t \cdot x_1 + t \cdot x_2 + \ldots + t \cdot x_n$ for some $t \in \{1, 2, 3, \ldots\}$.*

Theorem 8.10. *Suppose $\sigma \subseteq A^2$ is a congruence, $\rho(x_1, x_2, y_1, y_2)$ is a bridge from σ to σ such that $\rho(x, x, y, y)$ defines a full relation, $\mathrm{pr}_{1,2}(\rho) = \omega$, ω is a minimal relation compatible with σ such that $\omega \supsetneq \sigma$. Then there exists a prime number p and a relation $\zeta \subseteq A \times A \times \mathbb{Z}_p$ such that $(x_1, x_2, 0) \in \zeta \Leftrightarrow (x_1, x_2) \in \sigma$ and $\mathrm{pr}_{1,2}\, \zeta = \omega$.*

Proof. Since the relations ρ and ω are compatible with σ, we consider A/σ instead of A and assume that σ is the equality relation, ρ and ω are relations on A/σ.

Without loss of generality we assume that $\rho(x_1, x_2, y_1, y_2) = \rho(y_1, y_2, x_1, x_2)$ and $(a, b, a, b) \in \rho$ for any $(a, b) \in \omega$. Otherwise, we consider the relation ρ' instead of ρ, where

$$\rho'(x_1, x_2, y_1, y_2) = \exists z_1 \exists z_2\; \rho(x_1, x_2, z_1, z_2) \wedge \rho(y_1, y_2, z_1, z_2).$$

We prove by induction on the size of A. Assume that for some subuniverse $A' \subsetneq A$ we have $(A' \times A') \cap (\omega \setminus \sigma) \neq \varnothing$. By σ' we denote the restriction of σ to A'. By ω' we denote a minimal relation compatible with σ' such that $\sigma' \subsetneq \omega' \subseteq (A' \times A') \cap \omega$. By the inductive assumption for $\rho \cap (\omega' \times \omega')$ there exists a relation $\zeta' \subseteq A' \times A' \times \mathbb{Z}_p$ such that $(x_1, x_2, 0) \in \zeta' \Leftrightarrow (x_1, x_2) \in \sigma'$ and $\mathrm{pr}_{1,2}(\zeta') = \omega'$. Put

$$\zeta(x_1, x_2, z) = \exists y_1 \exists y_2\; \rho(x_1, x_2, y_1, y_2) \wedge \zeta'(y_1, y_2, z).$$

It is easy to see that ζ satisfies the necessary conditions.

Thus, we assume that for any subuniverse $A' \subsetneq A$ we have $(A' \times A') \cap (\omega \setminus \sigma) = \varnothing$.

Consider a pair $(a_1, a_2) \in \omega \setminus \sigma$. Then $\{a \mid (a_1, a) \in \omega\} = \{a \mid (a, a_2) \in \omega\} = A$. Hence, any element connected in ω to some other element is connected to all elements. Since $(a_1, a), (a, a_2) \in \omega$ for every $a \in A \setminus \{a_1, a_2\}$, if $|A| > 2$ then $\omega = A \times A$.

If $|A| = 2$ and $\omega \neq A \times A$ then $\omega = \{(a, a), (a, b), (b, b)\}$. This case cannot happen because the corresponding relation ρ is not preserved by any idempotent WNU.

Thus, we assume that $\omega = A \times A$.

Let us show that for any $a_1, a_2, a_3 \in A$ there exists a unique a_4 such that $(a_1, a_2, a_3, a_4) \in \rho$. For every $a \in A$ put $\lambda_a(x_1, x_2) = \exists y_2 \rho(x_1, x_2, a, y_2)$. It is easy to see that $\sigma \subsetneq \lambda_a \subseteq \omega$. Therefore $\lambda_a = \omega = A \times A$ for every a. We consider the unary relation defined by $\delta(x) = \rho(a_1, a_2, a_3, x)$. By the above fact δ is not empty. If δ contains more than one element, then we get a contradiction with the fact that there are no proper subuniverses.

Then ρ is a strongly rich relation. By Theorem 8.8, there exist an Abelian group $(A;+)$ and bijective mappings $\phi_1, \phi_2, \phi_3, \phi_4 \colon A \to A$ such that

$$\rho = \{(a_1, a_2, b_1, b_2) \mid \phi_1(a_1) + \phi_2(a_2) + \phi_3(b_1) + \phi_4(b_2) = 0\}.$$

We know that $(a, a, b, b) \in \rho$ for any $a, b \in A$, $\rho(x_1, x_2, y_1, y_2) = \rho(y_1, y_2, x_1, x_2)$. Then without loss of generality we can assume that $\phi_1(x) = \phi_3(x) = x$, $\phi_2(x) = \phi_4(x) = -x$.

Since w is a special WNU, it follows from Lemma 8.9 that w on A is defined by $x_1 + \ldots + x_m$. Therefore, the relation $\zeta \subseteq A \times A \times A$ defined by $\zeta = \{(b_1, b_2, b_3) \mid b_1 - b_2 + b_3 = 0\}$ is preserved by w. If $(A;+)$ is not simple, then there exists a subuniverse $A' \subsetneq A$ contradicting our assumption. Therefore, $(A;+)$ is a simple Abelian group. □

Corollary 8.10.1. *Suppose $\sigma \subseteq A^2$ is an irreducible congruence, $\rho(x_1, x_2, y_1, y_2)$ is a bridge from σ to σ such that $\rho(x, x, y, y)$ defines a full relation. Then there exists a prime number p and a relation $\zeta \subseteq A \times A \times \mathbb{Z}_p$ such that $(x_1, x_2, 0) \in \zeta \Leftrightarrow (x_1, x_2) \in \sigma$ and $\mathrm{pr}_{1,2}\,\zeta = \sigma^*$.*

Lemma 8.11. *Suppose ρ is an optimal bridge from σ_1 to σ_2, σ_1 and σ_2 are different irreducible congruences. Then $\widetilde{\rho} \supsetneq \sigma_2$, where $\widetilde{\rho}(x, y) = \rho(x, x, y, y)$.*

Proof. It is easy to see that $\sigma_1 \subseteq \widetilde{\rho}$ and $\sigma_2 \subseteq \widetilde{\rho}$. Therefore, if $\sigma_1 \not\subseteq \sigma_2$, then $\widetilde{\rho} \supsetneq \sigma_2$.

Assume that $\sigma_1 \subsetneq \sigma_2$. Assume the contrary, that is, $\rho(x, x, y, y) = \sigma_2(x, y)$.

First, we want to get the following property: for every $(a, b, c, d) \in \rho$ we have $(a, d) \in \sigma_2$. Put $\rho_1(x_1, x_2, y_1, y_2) = \rho(x_1, x_2, y_1, y_2) \wedge \sigma_2(x_1, y_2)$. If ρ_1 is a bridge then we replace ρ by ρ_1. Assume that ρ_1 is not a bridge, then for every $(a, b, c, d) \in \rho$ with $(a, d) \in \sigma_2$ we have $(a, b) \in \sigma_1$. Put $\rho_2(x_1, x_2, y_1, y_2) = \exists z\ \rho(x_1, x_2, z, y_1) \wedge \sigma_2(x_1, y_2)$ and replace ρ by ρ_2.

Second, we replace ρ by the relation defined by $\rho(x_1, x_2, y_1, y_2) \wedge \sigma_1^*(x_1, x_2) \wedge \sigma_2^*(y_1, y_2)$, which has the same properties.

Let D_0 be the domain of the congruences σ_1 and σ_2. We build a sequence of subsets $D_0, D_1, \ldots, D_s$ (not necessarily subuniverses) such that for every i there exists a unary operation $h_i : D_0 \to D_0$ such that $h_i(h_i(x)) = h_i(x)$, $h_i(D_0) = D_i$, and h_i preserves the relation ρ. It is not hard to see that $h_i(w(x_1, \ldots, x_m))$ is an idempotent WNU on D_i. By w_i we denote a special WNU on D_i that can be derived from the idempotent WNU on D_i. For any relation δ and any formula Θ by $\delta^{(i)}$ and $\Theta^{(i)}$ we denote their restriction to D_i. We require $\rho^{(i)}$ to be a reflexive bridge from $\sigma_1^{(i)}$ to $\sigma_2^{(i)}$ for every i.

Suppose we have a sequence $D_0, D_1, \ldots, D_s$. First, we want to show that for any $(b_1, b_2) \in (\sigma_2^*)^{(s)} \setminus \sigma_2^{(s)}$ the unary operation $g(x) = w_s(b_1, \ldots, b_1, x)$ maps a bridge $\rho^{(s)}$ to a bridge. To prove this, we need to show that $g(\rho^{(s)})$ contains a tuple (d_1, d_2, e_1, e_2) with $(d_1, d_2) \notin \sigma_1$. We know that there exists $(a_1, a_2, b_1, b_2) \in \rho$. Since $(a_1, b_2) \in \sigma_2$, we have $(a_1, b_1) \notin \sigma_2$. Since σ_2 is irreducible, $\mathrm{pr}_{1,3}(\rho)$ contains (b_1, b_2). Since $\rho(x, x, y, y) = \sigma_2(x, y)$, there exists $(b_1, b_1', b_2, b_2') \in \rho^{(s)}$ such that $(b_1, b_1') \notin \sigma_1$. Restrict w_s to the equivalence class of $\sigma_2^{(s)}$ containing b_1. The obtained operation and the equivalence class we denote by w' and E, correspondingly. Put $\rho' = \rho \cap (E^2 \times D_s^2)$ and $\sigma_1' = \sigma_1^{(s)} \cap E^2$. Let $\omega \subseteq \sigma_1^* \cap E^2$ be a minimal relation compatible with σ_1' such that $\omega \supsetneq \sigma_1'$. It is not hard to check that the formula

$$\exists y_1 \exists y_2\ \rho'(x_1, x_2, y_1, y_2) \wedge \rho'(x_1', x_2', y_1, y_2) \wedge \omega(x_1, x_2) \wedge \omega(x_1', x_2')$$

defines a reflexive bridge $\rho''(x_1, x_2, x_1', x_2')$ from σ_1' to σ_1'. By Theorem 8.10, there exists a prime number p and a relation $\zeta \subseteq E \times E \times \mathbb{Z}_p$ such that $(x_1, x_2, 0) \in \zeta \Leftrightarrow (x_1, x_2) \in \sigma_1'$ and $\mathrm{pr}_{1,2}\,\zeta = \omega$. Therefore, for some $(e_1, e_2) \in \omega$ we have $(w_s(b_1, \ldots, b_1, e_1), w_s(b_1, \ldots, b_1, e_2)) \notin \sigma_1$ and the unary operation $g(x) = w_s(b_1, \ldots, b_1, x)$ maps the bridge $\rho^{(s)}$ to a bridge.

Consider two cases. Case 1: there exists $(b_1, b_2) \in (\sigma_2^*)^{(s)} \setminus \sigma_2^{(s)}$ such that $w_s(b_1, \ldots, b_1, x) \neq x$ on $D^{(s)}$. Then put $h_{s+1}(x) = w_s(b_1, \ldots, b_1, h_s(x))$ and $D_{s+1} = h_{s+1}(D_0)$. Thus, we made our sequence longer.

Case 2: For any $(b_1, b_2) \in (\sigma_2^*)^{(s)} \setminus \sigma_2^{(s)}$ we have $w_s(b_1, \ldots, b_1, x) = x$ on D_s. Assume that there exist $(a_1, a_2, a_3, a_4), (a_1', a_2, a_3, a_4) \in \rho^{(s)}$ such that $(a_1, a_1') \notin \sigma_1$. Since w_s preserves $\rho^{(s)}$, we have

$$(w_s(a_1', a_1, \ldots, a_1), w_s(a_2, \ldots, a_2, a_1), w_s(a_3, \ldots, a_3, a_1), w_s(a_4, \ldots, a_4, a_1)) \in \rho^{(s)}.$$

Since $(a_1, a_4) \in \sigma_2$ and $(a_1, a_2), (a_1', a_2) \in \sigma_2$, we have $(a_1, a_3), (a_2, a_3), (a_3, a_4) \in (\sigma_2^*)^{(s)} \setminus \sigma_2^{(s)}$. Therefore, the above tuple equals (a_1', a_1, a_1, a_1) and belongs to $\rho^{(s)}$, which contradicts our assumptions. In the same way we can show that $(a_1, a_2, a_3, a_4), (a_1, a_2, a_3', a_4) \in \rho^{(s)}$ implies $(a_3, a_3') \in \sigma_2$.

Consider $(a_1, a_2, b_1, b_2) \in \rho^{(s)}$ with $(b_1, b_2) \notin \sigma_2$ and the formula

$$\Theta = \rho(z, x_1, x_2, x_3) \wedge \rho(z', x_1, x_2', x_3') \wedge \rho(z, x_4, x_5, x_6) \wedge \rho(z', x_4, x_5', x_6').$$

Suppose $\Theta(x_2, x_2', x_5, x_5')$ defines a relation ϵ. Since $h_s(h_s(x)) = h_s(x)$ and h_s preserves ρ, the formula $\Theta^{(s)}(x_2, x_2', x_5, x_5')$ defines the relation $\epsilon^{(s)}$. By sending (z, x_1, x_2, x_3) to (a_1, a_2, b_1, b_2), (z', x_1, x_2', x_3') to (a_2, a_2, a_2, a_2), (z, x_4, x_5, x_6) to (a_1, a_2, b_1, b_2), (z', x_4, x_5', x_6') to (a_2, a_2, a_2, a_2), we show that $(b_1, a_2, b_1, a_2) \in \epsilon$. Assume that $\epsilon^{(s)}$ is not a bridge, then there exists $(a, a, b, c) \in \epsilon^{(s)}$ such that $(b, c) \notin \sigma_2$. This contradicts the rectangularity of the first and third variables of $\rho^{(s)}$.

Let us show that ϵ is also a bridge. Assume the contrary. Then without loss of generality we assume that there exists $(d_0, d_0, d_1, d_2) \in \epsilon$ such that $(d_1, d_2) \notin \sigma_2$. Put $\delta_0(y, z) = \exists x\ \epsilon(x, x, y, z)$. Since σ_2 is irreducible, we have $(b_1, b_2) \in \delta_0$ and there exists d such that $(d, d, b_1, b_2) \in \epsilon$, which means that $(h_s(d), h_s(d), b_1, b_2) \in \epsilon^{(s)}$. This contradiction proves that ϵ is a bridge. By sending $(z, x_1, x_2, x_2', x_3, x_3')$ to $(a_1, a_2, b_1, b_1, b_2, b_2)$ and $(z', x_4, x_5, x_5', x_6, x_6')$ to $(a_1, a_1, a_1, a_1, a_1, a_1)$ we can show that $(b_1, b_1, a_1, a_1) \in \epsilon$. Since we can compose the bridges ρ and ϵ, we get a contradiction with the fact that ρ is optimal. □

8.4 Previous reductions

Theorem 8.12. *Suppose $D^{(0)}, D^{(1)}, \ldots, D^{(s)}$ is a strategy for Ω, the solution set of $\Omega^{(i)}$ is subdirect for every $i \in \{0, 1, \ldots, s\}$, $j < s$, $D^{(s+1)}$ is a proper reduction, at least one of the two reductions $D^{(j+1)}$, $D^{(s+1)}$ is nonlinear, $(\Omega^{(j)}(x_1, \ldots, x_n))^{(s+1)}$ defines a nonempty relation. Then $(\Omega^{(j+1)}(x_1, \ldots, x_n))^{(s+1)}$ defines a nonempty relation.*

Proof. Assume the contrary. Let $\{x_1, \ldots, x_n, y_1, \ldots, y_t\}$ be the set of all variables appearing in Ω. Suppose $\Omega^{(j)}(x_1, \ldots, x_n, y_1, \ldots, y_t)$ defines the relation ρ. We consider the type of the reduction $D^{(j+1)}$ and the type of the reduction $D^{(s+1)}$.

$D^{(s+1)}$ **is an absorbing reduction.** Since $\Omega^{(s)}(x_1, \ldots, x_n, y_1, \ldots, y_t)$ defines a subdirect relation, Lemma 7.5 implies that $\Omega^{(s+1)}(x_1, \ldots, x_n, y_1, \ldots, y_t)$ defines a nonempty relation.

$D^{(j+1)}$ **is a PC reduction.** First, using the definition of a PC subuniverse, for every variable y_i we choose a congruence on $D_{y_i}^{(j)}$ such that $D_{y_i}^{(j+1)}$ is an equivalence class of this congruence. Second, we factorize the variables $y_1, \ldots, y_t$ of ρ by these congruences and replace these variables by a set of PC variables. As a result we get a relation $\rho'(x_1, \ldots, x_n, z_1, \ldots, z_k)$, where the domain of z_i is a PC algebra for every i. By ρ_l we denote the relation obtained from ρ' by restricting of the variables $x_1, \ldots, x_n$ to $D^{(l)}$. Obviously, $\rho_j = \rho'$.

Since ρ is subdirect, every variable z_i takes on all values in ρ_j. Let us prove by induction on $l \in \{j, \ldots, s\}$ that the variable z_i either takes on all values in ρ_l, or just one value. Let l be the minimal number such that this is not true. Then z_i takes on all values in ρ_{l-1}.

Let us consider the type of the reduction $D^{(l)}$. If it is an absorbing or central reduction, then by Corollaries 7.1.1, 7.6.1 we get a center or a binary absorbing set on the domain of z_i and therefore on the domain $D_{y_q}^{(j)}$ for some variable y_q. This contradicts the fact that $D^{(j+1)}$ is a PC reduction. Similarly, if it is a PC or linear reduction then we get a contradiction with Corollaries 7.11.1 and 7.15.1 correspondingly. Thus, we know that every variable z_i of ρ_s either takes on all values, or just one value.

Assume that $D^{(s+1)}$ is a central reduction or a linear reduction. Let 0 be the value in the domain of every variable z_i corresponding to the reduction $D^{(j+1)}$. Let us consider the tuple $(a_1, \ldots, a_n, b_1, \ldots, b_k) \in \rho_s$ with the maximal number of 0s such that $a_1, \ldots, a_n \in D^{(s+1)}$. Without loss of generality assume that $b_i = 0$ for every $i \in \{k' + 1, \ldots, k\}$. Then we consider the relation ρ_s' defined by $\rho_s(x_1, \ldots, x_n, z_1, \ldots, z_{k'}, 0, \ldots, 0)$. Since $\rho^{(s)}$ is subdirect, every variable z_i takes on value 0 in ρ_s'. Therefore, $z_1, \ldots, z_{k'}$ take on all values from their domains in ρ'_s. By Corollaries 7.6.1, 7.15.1, if we restrict variables $x_1,\dots,x_n$ of ρ'_s to $D^{(s+1)}$, then we restrict the remaining variables of ρ'_s to a center or to a linear subuniverse. Hence, we get a center or a linear subuniverse on the domain of z_i. This contradicts the fact that $D^{(j+1)}$ is a PC reduction.

Assume that $D^{(s+1)}$ is a PC reduction. We factorize variables $x_1,\dots,x_n$ of ρ_s by $(s+1)$ and replace these variables by PC variables $u_1,\dots,u_h$. As a result we get a relation $\rho''(u_1,\dots,u_h,z_1,\dots,z_k)$. By Lemma 7.11, ρ'' can be represented as a conjunction of binary relations with the parallelogram property. Since $\rho^{(s)}$ is subdirect, we can not have a binary relation involving a variable from $\{u_1,\dots,u_h\}$ and a variable from $\{z_1,\dots,z_k\}$. This means that if we put $z_i=0$ for every i we do not reduce the projection of ρ'' onto the first h variables. This contradicts our assumption.

$D^{(s+1)}$ is a central reduction, $D^{(j+1)}$ is an absorbing or central reduction. Let N be the maximal number such that there exists a tuple in ρ with the first N elements from $D^{(s+1)}$ and the last t elements from $D^{(j+1)}$. Let the relation ρ' be obtained from ρ by restriction of the first n variables to $D^{(s)}$. We consider the relation ρ' as a ternary relation $\rho'\subseteq(D^{(s)}_{x_1}\times\dots\times D^{(s)}_{x_N})\times(D^{(s)}_{x_{N+1}}\times\dots\times D^{(s)}_{x_n})\times(D^{(j)}_{y_1}\times\dots\times D^{(j)}_{y_t})$, where $D^{(s+1)}_{x_1}\times\dots\times D^{(s+1)}_{x_N}$ is a center of $D^{(s)}_{x_1}\times\dots\times D^{(s)}_{x_N}$, $D^{(s+1)}_{x_{N+1}}\times D^{(s)}_{x_{N+2}}\times\dots\times D^{(s)}_{x_n}$ is a center of $D^{(s)}_{x_{N+1}}\times D^{(s)}_{x_{N+2}}\times\dots\times D^{(s)}_{x_n}$, and $D^{(j+1)}_{y_1}\times\dots\times D^{(j+1)}_{y_t}$ is a center or a binary absorbing set in $D^{(j)}_{y_1}\times\dots\times D^{(j)}_{y_t}$. This gives us a contradiction with Corollary 7.9.1.

$D^{(s+1)}$ is a central reduction, $D^{(j+1)}$ is a linear reduction. We factorize the last t variables of ρ by $(j+1)$ and restrict the first n variables to $D^{(s)}$. As a result we get a relation $\rho'(x_1,\dots,x_n,z_1,\dots,z_t)$, where the domain of z_i is a linear algebra for every i. By Corollary 7.6.1, if we restrict variables $x_1,\dots,x_n$ of ρ' to $D^{(s+1)}$ then we restrict the remaining variables to a center, which is not possible for a linear algebra.

$D^{(s+1)}$ is a linear reduction, $D^{(j+1)}$ is an absorbing or central reduction. By ρ' we denote the relation obtained from ρ by the restriction of the variables $x_1,\dots,x_n$ to $D^{(s)}$ and factorization of them by $(s+1)$. By Corollaries 7.1.1, 7.6.1, if we restrict the variables $y_1,\dots,y_t$ of ρ' to $D^{(j+1)}$, then we get a restriction of the remaining variables to a center or a binary absorbing subuniverse, which is not possible for linear algebra.

$D^{(s+1)}$ is a PC reduction, $D^{(j+1)}$ is an absorbing, central, or linear reduction. Again, using the definition of a PC subuniverse, for every variable x_i we choose a congruence on $D^{(s)}_{x_i}$ such that $D^{(s+1)}_{x_i}$ is an equivalence class of this congruence. Then, we factorize the variables $x_1,\dots,x_n$ of ρ by these congruences and replace these variables by a set of PC variables. As a result we get a relation $\rho'(z_1,\dots,z_k,y_1,\dots,y_t)$, where the domain of z_i is a PC algebra for every i. Let 0 be the value in the domain of every variable z_i corresponding to the reduction $D^{(s+1)}$. Let us consider a tuple from ρ' with the maximal number of elements equal to 0 and the last t elements from $D^{(j+1)}$. Without loss of generality assume that this tuple is $(a_1,\dots,a_{k'},0,\dots,0,b_1,\dots,b_t)$, where $a_i\neq 0$ for every $i\in\{1,\dots,k'\}$. Let us consider the relation $\rho''(z_1,\dots,z_{k'},y_1,\dots,y_t)$ defined by $\rho'(z_1,\dots,z_{k'},0,\dots,0,y_1,\dots,y_t)$. It is easy to see that every variable z_i takes on all values in ρ''. By Corollaries 7.1.1, 7.6.1, 7.15.1, the restriction of $y_1,\dots,y_t$ to $D^{(j+1)}$ implies the restriction of each of the variables $z_1,\dots,z_{k'}$ to a binary absorbing subuniverse, a center, or a linear subuniverse. This contradicts the definition of a PC reduction. □

Corollary 8.12.1. *Suppose Θ is a cycle-consistent CSP instance, $D^{(0)},D^{(1)},\dots,D^{(s)}$ is a strategy for Θ, $\Upsilon\in\mathrm{ExpCov}(\Theta)$ is a tree-formula, x is a parent of x_1 and x_2, B is a center of $D^{(s)}_x$, or B is a PC subuniverse of $D^{(s)}_x$ and $D^{(s)}_y$ has no binary absorption and center for every y. Then the pp-formula $\Upsilon^{(s)}(x_1,x_2)$ defines a binary relation with a nonempty intersection with $B\times B$.*

Lemma 8.13. *Suppose $D^{(0)}, D^{(1)}, \dots, D^{(s)}$ is a strategy for the constraint $\rho(x_1, \dots, x_n)$, $D^{(s+1)}$ is a linear reduction,*

$$(b_1, \dots, b_t, a_{t+1}, \dots, a_n) \in \rho,$$
$$(a_1, \dots, a_t, b_{t+1}, \dots, b_n) \in \rho,$$
$$(b_1, \dots, b_t, b_{t+1}, \dots, b_n) \in \rho,$$
$$(a_1, \dots, a_t, a_{t+1}, \dots, a_n) \in D^{(s+1)}.$$

Then there exists $(d_1, d_2, \dots, d_n) \in \rho^{(s+1)}$.

Proof. Let I be the set of all i such that $D^{(0)}_{x_i}$ is not linear. We prove by induction on the sum $\sum_{i\in I} |D^{(0)}_{x_i}|$.

If $s = 0$, then we put $b'_i = w(b_i, \dots, b_i, a_i)$ for every i. Obviously, $(b'_1, \dots, b'_n) \in \rho^{(s+1)}$.

From here on we assume that $s > 0$. Put

$$\Omega = \rho(y_1, \dots, y_t, x_{t+1}, \dots, x_n) \wedge \rho(x_1, \dots, x_t, y_{t+1}, \dots, y_n) \wedge \rho(y_1, \dots, y_t, y_{t+1}, \dots, y_n).$$

Since $\rho^{(j)}$ is subdirect, the solution set of $\Omega^{(j)}$ is also subdirect for every $j \in \{0, 1, \dots, s\}$.

If $(\Omega^{(1)}(x_1, \dots, x_n))^{(s+1)}$ is not empty, then we can apply the inductive assumption to $\rho^{(1)}$ to complete the proof.

Suppose $(\Omega^{(1)}(x_1, \dots, x_n))^{(s+1)}$ is empty. Since $(\Omega(x_1, \dots, x_n))^{(s+1)}$ is not empty, if $D^{(1)}$ is a nonlinear reduction then we get a contradiction with Theorem 8.12.

It remains to consider the case when $D^{(1)}$ is linear. Suppose $\Omega(x_1, \dots, x_n, y_1, \dots, y_n)$ defines the relation ρ'. By ρ'' we denote the relation obtained from ρ' by factorization of $y_1, \dots, y_n$ by (1). For an element b by $b^{(1)}$ we denote the equivalent class containing b.

We consider two cases. Case 1. There does not exist a tuple $(c_1, \dots, c_n, d_1, \dots, d_n) \in \rho'$ such that $c_1, \dots, c_n \in D^{(s+1)}$ and $d_1, \dots, d_t \in D^{(1)}$ (we do not require $d_{t+1}, \dots, d_n$ to be in $D^{(1)}$). Put

$$\epsilon(x_1, \dots, x_n, z_1, \dots, z_t) = \exists z_{t+1} \dots \exists z_n \; \rho''(x_1, \dots, x_n, z_1, \dots, z_n).$$

Put $b'_i = w(b_i, \dots, b_i, a_i)$. Since $D^{(1)}$ is a linear reduction and

$$(a_1, \dots, a_n, b^{(1)}_1, \dots, b^{(1)}_t), (b_1, \dots, b_n, a^{(1)}_1, \dots, a^{(1)}_t), (b_1, \dots, b_n, b^{(1)}_1, \dots, b^{(1)}_t) \in \epsilon,$$

we have

$$(b'_1, \dots, b'_n, a^{(1)}_1, \dots, a^{(1)}_t), (b'_1, \dots, b'_n, b^{(1)}_1, \dots, b^{(1)}_t) \in \epsilon$$

We know that $(b'_1, \dots, b'_n) \in D^{(1)}$. Let us restrict the variables $x_1, \dots, x_n$ of the relation ϵ to $D^{(1)}$. The obtained relation we denote by ϵ'. It is easy to find a strategy $E^{(1)}, \dots, E^{(s)}$ for $\epsilon'(x_1, \dots, x_n, z_1, \dots, z_t)$ such that $E^{(j)}_{x_i} = D^{(j)}_{x_i}$ and $E^{(j)}_{z_i}$ contains $\{a^{(1)}_i\}$ for every i and j. By $E^{(s+1)}$ we denote the reduction of $E^{(s)}$ such that $E^{(s+1)}_{x_i} = D^{(s+1)}_{x_i}$ and $E^{(s+1)}_{z_i} = \{a^{(1)}_i\}$ for every i. Then we apply the inductive assumption for ϵ' and a strategy $E^{(1)}, \dots, E^{(s)}$ to get a tuple in $\epsilon'^{(s+1)}$. This contradicts our assumption.

Case 2: There exists a tuple $(c_1, \dots, c_n, d_1, \dots, d_n) \in \rho'$ such that $c_1, \dots, c_n \in D^{(s+1)}$ and $d_1, \dots, d_t \in D^{(1)}$. Then $d^{(1)}_i = a^{(1)}_i$ for $i \in \{1, \dots, t\}$. Since $c_1, \dots, c_n \in D^{(s+1)}$, we have $c^{(1)}_i = a^{(1)}_i$ for every $i \in \{1, \dots, n\}$. Therefore,

$$(d_1, \dots, d_t, c_{t+1}, \dots, c_n, a^{(1)}_1, \dots, a^{(1)}_t, a^{(1)}_{t+1}, \dots, a^{(1)}_n) \in \rho'',$$
$$(c_1, \dots, c_t, c_{t+1}, \dots, c_n, a^{(1)}_1, \dots, a^{(1)}_t, d^{(1)}_{t+1}, \dots, d^{(1)}_n) \in \rho'',$$
$$(d_1, \dots, d_t, c_{t+1}, \dots, c_n, a^{(1)}_1, \dots, a^{(1)}_t, d^{(1)}_{t+1}, \dots, d^{(1)}_n) \in \rho''.$$

Let us restrict the variables $x_1, \dots, x_n$ of the relation ρ'' to $D^{(1)}$. The obtained relation we denote by ϵ. In the same way as in case 1 we define a strategy $E^{(1)}, \dots, E^{(s)}$ for $\epsilon(x_1, \dots, x_n, z_1, \dots, z_n)$ and a reduction $E^{(s+1)}$ such that $E_{x_i}^{(j)} = D_{x_i}^{(j)}$ for every i and j. Then we apply the inductive assumption for ϵ and a strategy $E^{(1)}, \dots, E^{(s)}$ to get a tuple in $\epsilon^{(s+1)}$. This contradicts our assumption that $(\Omega^{(1)}(x_1, \dots, x_n))^{(s+1)}$ is empty. □

8.5 Existence of a bridge

In this subsection we explain how to get a bridge from a rectangular relation and to compose bridges appearing in the instance.

Lemma 8.14. *Suppose $\rho \subseteq A_1 \times \cdots \times A_n$ is a subdirect relation, the first and the last variables of ρ are rectangular, there exist $(b_1, a_2, \dots, a_n), (a_1, \dots, a_{n-1}, b_n) \in \rho$ such that $(a_1, a_2, \dots, a_n) \notin \rho$. Then there exists a bridge δ from $\mathrm{Con}(\rho, 1)$ to $\mathrm{Con}(\rho, n)$ such that $\delta(x, x, y, y)$ is equal to the projection of ρ onto the first and the last variables.*

Proof. The required bridge can be defined by

$$\delta(x_1, x_2, y_1, y_2) = \exists z_2 \dots \exists z_{n-1}\ \rho(x_1, z_2, \dots, z_{n-1}, y_1) \wedge \rho(x_2, z_2, \dots, z_{n-1}, y_2).$$

□

Theorem 8.15. *Suppose Θ is a cycle-consistent connected formula such that every constraint relation is a critical rectangular relation. Then for every constraints C, C' with the corresponding variables x, x' there exists a bridge δ from $\mathrm{Con}(C, x)$ to $\mathrm{Con}(C', x)$ such that $\delta(x, x, y, y)$ contains all pairs of elements linked in Θ. Moreover, if $\mathrm{Con}(C'', x'') \neq \mathrm{LinkedCon}(\Theta, x'')$ for some constraint $C'' \in \Theta$ and a variable x'', then $\delta(x, x, y, y)$ contains all pairs of elements linked in Θ', where Θ' is obtained from Θ by replacement of every constraint relation by its cover.*

Proof. Since C and C' are connected, there exists a path $z_0 C_1 z_1 C_2 z_2 \dots C_{t-1} z_{t-1} C_t z_t$, where $z_0 = x$, $z_t = x'$, $C_1 = C$, $C_t = C'$, and C_i and C_{i+1} are adjacent in z_i for every i.

By Lemma 8.3, every relation defined by $\mathrm{Con}(C_0, x_0)$ for some C_0 and x_0 is an irreducible congruence. Suppose σ_i is an optimal bridge from $\mathrm{Con}(C_i, z_i)$ to $\mathrm{Con}(C_{i+1}, z_i)$, δ_i is a bridge from $\mathrm{Con}(C_i, z_{i-1})$ to $\mathrm{Con}(C_i, z_i)$ from Lemma 8.14 for every i. Then we compose all bridges together and define a new bridge $\delta(u_0, u_0', v_t, v_t')$ by

$$\exists u_1 \exists u_1' \exists v_1 \exists v_1' \dots \exists u_{t-1} \exists u_{t-1}' \exists v_{t-1} \exists v_{t-1}'\ \delta_1(u_0, u_0', v_1, v_1') \wedge \bigwedge_{i=1}^{t-1} (\sigma_i(v_i, v_i', u_i, u_i') \wedge \delta_{i+1}(u_i, u_i', v_{i+1}, v_{i+1}')). \quad (5)$$

Since Θ is cycle-consistent, if $x = x'$ then δ is a reflexive bridge from $\mathrm{Con}(C, x)$ to $\mathrm{Con}(C', x)$. Thus we proved that any two constraints with a common variable are adjacent.

Since the instance Θ is cycle-consistent, there exists a path in Θ starting at x and ending at x' that connects any pair of elements linked in Θ. Since every pair of constraints with common variable are adjacent, we can assume that the above path $z_0 C_1 z_1 C_2 z_2 \dots C_{t-1} z_{t-1} C_t z_t$ satisfies this property. Then it is easy to check that $\delta(x, x, y, y)$ contains all pairs of elements linked in Θ.

To prove the remaining part of the theorem, assume that $\mathrm{Con}(C'', x'') \neq \mathrm{LinkedCon}(\Theta, x'')$ for some constraint $C'' \in \Theta$ and a variable x''. For any bridge ρ, by $\widetilde{\rho}$ we denote binary relation defined by $\rho(x, x, y, y)$. First, observe that any bridge ρ from σ_1 to σ_2 defined by Lemma 8.14 satisfies one of the following properties:

1. $\mathrm{Con}(\widetilde{\rho}, 1) = \sigma_1$ and $\mathrm{Con}(\widetilde{\rho}, 2) = \sigma_2$,
2. $\mathrm{Con}(\widetilde{\rho}, 1) \supsetneq \sigma_1$ and $\mathrm{Con}(\widetilde{\rho}, 2) \supsetneq \sigma_2$.

Similarly, one of the above properties holds for any reflexive bridge. Thus, every bridge in (5) satisfies one of the above properties.

By the first part of the theorem $\mathrm{Opt}(\mathrm{Con}(C'', x'')) \supsetneq \mathrm{Con}(C'', x'')$. It is not hard to see that if we join bridges together as in (5) and at least one of the bridges satisfies property 2 then the obtained bridge satisfies property 2. We may assume that any path goes through the variable x'', which guarantees that every bridge we obtain satisfies property 2. Thus, we showed that $\mathrm{Opt}(\mathrm{Con}(C_0, x_0)) \supsetneq \mathrm{Con}(C_0, x_0)$ for any constraint $C_0 \in \Theta$ and any variable x_0 in it.

To complete the proof, notice that the composition of bridges δ_i and σ_i in (5) gives a bridge ρ_i such that $\rho_i(x, x, y, y)$ contains the projection of the cover of C_i onto the variables z_{i-1} and z_i. □

Corollary 8.15.1. *Suppose Θ is a cycle-consistent connected formula such that every constraint relation is a critical rectangular relation. Then for every constraints C, C' with a common variable x there exists a bridge δ from $\mathrm{Con}(C, x)$ to $\mathrm{Con}(C', x)$ such that $\delta(x, x, y, y)$ contains the relation $\mathrm{LinkedCon}(\Theta, x)$.*

8.6 Growing population divides into colonies.

In this section we prove a theorem that clarifies the inductive strategy used in the proof of Theorem 9.8. To simplify explanation we decided to avoid our usual terminology. Instead, we argue in terms of organisms, reproduction, and friendship.

We consider a set X whose elements we call *organisms*. At every moment some organisms give a birth to new organisms, as a result we get a sequence of organisms $X_1 \subseteq X_2 \subseteq X_3 \subseteq \dots$, where $X_i \subseteq X$ and $|X_i| < \infty$ for every i. Note that every organism has only one parent.

Every organism has a characteristic that we call *strength*, that is a mapping $\xi : X \to \{1, 2, \dots, S\}$. Also we have a binary reflexive symmetric relation F on the set X, which we call *friendship*. For an organism x by $\mathrm{BirthDate}(x)$ we denote the minimal i such that $x \in X_i$. A sequence of organisms $x_1, \dots, x_n$ such that x_i is a friend of x_{i+1} for every i is called *a path*.

Theorem 8.16. *Suppose $X_1, X_2, X_3, \dots$, ξ, and F satisfy the following conditions:*

1. ***A child is always weaker than its parent.*** *If y is the parent of x, then $\xi(y) > \xi(x)$.*
2. ***Older friends are parents's friends.*** *If $\mathrm{BirthDate}(y) < \mathrm{BirthDate}(x)$ and x is a friend of y, then the parent of x is a friend of y. Also a child and its parent can be friends.*
3. ***Friends's kids can be friends.*** *If $\mathrm{BirthDate}(x) = \mathrm{BirthDate}(y)$ and x is a friend of y, then the parents of x and y are friends.*
4. ***No one can have infinitely many friends.*** *$|\{y \in X \mid (x, y) \in F\}| < \infty$ for every $x \in X$.*
5. ***Reproduction never stops.*** *$|\bigcup_i X_i| = \infty$.*

Then there exists N such that X_N can be divided into two nonempty disjoint sets X'_N and X''_N such that no friendship between X'_N and X''_N.

Proof. Choose a maximal strength s such that we have infinitely many organisms of strength s. Then infinitely many of them have the same parent, hence, there exists a parent reproducing infinitely many times.

For every x and a strength s by $\text{KIDs}(x, s)$ we denote all children y of x such that there exists a path from x to y with all the variables in the path stronger than s. We consider the maximal s_0 such that $\text{KIDs}(x, s_0)$ is infinite for some variable x. Note that this implies that x is stronger than $s_0 + 1$.

By Y we denote the set of all organisms y such that there exists a path from x to y with all the variables in the path stronger than $s_0 + 1$. Note that Y includes x. Let us show that Y is finite. Assume the opposite. Let s be the maximal strength such that we have infinitely many organisms of this strength in Y. We consider all the parents of organisms from Y with strength s. It is not hard to see that all of them are also from Y. Since parents are stronger than children, we can have only finitely many of them. Therefore, there exists an organism z with infinitely many children from Y, which means that $\text{KIDs}(z, s_0 + 1)$ is infinite. This contradicts the maximality of s_0 and proves that Y is finite.

Let t be the moment such that all friends of organisms from Y get new friends before t. Consider an organism y from $\text{KIDs}(x, s_0)$ with $\text{BirthDate}(y) > t$. Choose a path from x to y in $X_{\text{BirthDate}(y)}$ with all organisms stronger than s_0. We consider the last organism u in the path such that $\text{BirthDate}(u) < \text{BirthDate}(y)$. Considering the moment $\text{BirthDate}(y) - 1$ we can show that there exists a path from x to u with all organisms but u stronger than $s_0 + 1$. Since u cannot get a new friend after the moment t we get a contradiction with the fact that u gets a new friend at the moment $\text{BirthDate}(y)$. □

9 Proof of the Main Theorems

9.1 Existence of a next reduction

Lemma 9.1. *Suppose $D^{(0)}, D^{(1)}, \ldots, D^{(s)}$ is a strategy for a cycle-consistent CSP instance Θ, $D^{(\top)}$ is a reduction of $\Theta^{(s)}$.*

1. *If there exists a 1-consistent reduction contained in $D^{(\top)}$ and $D^{(s+1)}$ is maximal among such reductions, then for every variable y of Θ there exists a tree-formula $\Upsilon_y \in \text{Coverings}(\Theta)$ such that $\Upsilon_y^{(\top)}(y)$ defines $D_y^{(s+1)}$.*

2. *Otherwise, there exists a tree-formula $\Upsilon \in \text{Coverings}(\Theta)$ such that $\Upsilon^{(\top)}$ has no solutions.*

Proof. The proof is based on the constraint propagation procedure. We consider the instance $\Theta^{(s)}$. We start with an empty set Υ_y for every y.

Then we introduce the recursive algorithm that gives a correct tree-formula Υ_y for every variable y. If at some step the obtained instance is 1-consistent, then we are done. Otherwise, we consider a constraint C that breaks 1-consistency. Then the current restrictions of the variables $z_1, \ldots, z_l$ in the constraint $C = \rho(z_1 \ldots, z_l)$ implies a stronger restriction of some variable z_i and the corresponding domain $D_{z_i}^{(s)}$. Then we change the tree-formula Υ_{z_i} describing the reduction of the variable z_i in the following way $\Upsilon_{z_i} := C \wedge \Upsilon_{z_1} \wedge \cdots \wedge \Upsilon_{z_l}$.

Note that we have to be careful with all the variables appearing in different Υ_y to avoid collisions. Every time we join Υ_u and Υ_v together we rename the variables so that they do not have common variables.

Obviously, this procedure either gives a maximal 1-consistent CSP instance whose domains are defined by tree-formulas Υ_y for every y, or it gives a contradiction, that is, a tree-formula that defines an empty-set, which can be taken as Υ. □

Theorem 9.2. *Suppose $D^{(0)}, D^{(1)}, \dots, D^{(s)}$ is a strategy for a cycle-consistent CSP instance Θ.*

- *If $D_x^{(s)}$ has a binary absorbing set B then there exists a 1-consistent absorbing reduction $D^{(s+1)}$ of $\Theta^{(s)}$ with $D_x^{(s+1)} \subseteq B$.*
- *If $D_x^{(s)}$ has a center B then there exists a 1-consistent central reduction $D^{(s+1)}$ of $\Theta^{(s)}$ with $D_x^{(s+1)} \subseteq B$.*
- *If $D_y^{(s)}$ has no binary absorption and center for every y but there exists a proper PC subuniverse B in $D_x^{(s)}$ for some x, then there exists a 1-consistent PC reduction $D^{(s+1)}$ of $\Theta^{(s)}$ with $D_x^{(s+1)} \subseteq B$.*

Proof. Without loss of generality we assume that B is a minimal center, minimal binary absorbing set, or minimal PC subuniverse. Let us reduce the domain $D_x^{(s)}$ to B. By Lemma 9.1, either we get a contradiction, or we get a 1-consistent reduction. We consider two cases. If we get a contradiction, then we consider the tree-formula Υ from Lemma 9.1. First, we consider the minimal set of variables $\{x_1, \dots, x_k\}$ from Υ whose parent is x such that $\Upsilon^{(s)}(x_1, \dots, x_k)$ does not have tuples in B^k. Since Θ is 1-consistent, $k \geqslant 2$. If B is a binary absorbing set, then we get a contradiction with Lemma 7.5. For other cases with $k = 2$ we get a contradiction from Corollary 8.12.1. If $k \geqslant 3$ and B is a center then we get a contradiction with Lemma 7.9.3. If $k \geqslant 3$ and B is a PC subuniverse then we get a contradiction with Corollary 7.11.2.

Thus, by Lemma 9.1, we have a 1-consistent reduction $D^{(\top)}$ of $\Theta^{(s)}$ such that for every variable y the new domain $D_y^{(\top)}$ can be defined by a tree-formula Υ_y. By Corollaries 7.1.1, 7.6.1, 7.11.1, for every y the domain $D_y^{(\top)}$ is a center, a binary absorbing set, or a PC subuniverse, correspondingly. □

Theorem 9.3. *Suppose $D^{(0)}, D^{(1)}, \dots, D^{(s)}$ is a strategy for a cycle-consistent CSP instance Θ, $D^{(\top)}$ is a nonlinear 1-consistent reduction. Then there exists a minimal 1-consistent reduction $D^{(s+1)}$ of the same type such that $D_x^{(s+1)} \subseteq D_x^{(\top)}$ for every variable x.*

Proof. Let us consider a minimal by inclusion 1-consistent reduction $D^{(s+1)}$ of $\Theta^{(s)}$ such that $D^{(s+1)}$ has the same type as $D^{(\top)}$ and $D_x^{(s+1)} \subseteq D_x^{(\top)}$ for every variable x.

Assume that for some z the domain $D_z^{(s+1)}$ is not a minimal center/binary absorbing set/PC subuniverse. Then choose a minimal center/binary absorbing set/PC subuniverse B of $D_z^{(s)}$ contained in $D_z^{(s+1)}$. We consider the reduction $D^{(\perp)}$ of $\Theta^{(s)}$ such that $D_z^{(\perp)} = B$, $D_y^{(\perp)} = D_y^{(s+1)}$ if $y \neq z$. Since $D_y^{(s+1)}$ is a minimal by inclusion reduction, Lemma 9.1 implies that there exists a tree-formula $\Upsilon \in \text{Coverings}(\Theta)$ such that $\Upsilon^{(\perp)}$ has no solutions. Again, we consider a minimal set of variables $\{z_1, \dots, z_k\}$ from Υ whose parent is z such that $\Upsilon^{(s+1)}(z_1, \dots, z_k)$ does not have tuples in B^k. Since $D_z^{(s+1)}$ is 1-consistent and $B \subsetneq D_z^{(s+1)}$, we have $k \geqslant 2$. If B is a binary absorbing set, then we get a contradiction with Lemma 7.5. If B is a center and $k = 2$, then we get a contradiction from Corollary 8.12.1. If $k \geqslant 3$ and B is a center then we get a contradiction with Lemma 7.9.3. It remains to consider the case when B is a PC subuniverse. Choose a minimal set of variables $y_1, \dots, y_t$ from Υ whose parent is not z such that $(\Upsilon^{(s)}(z_1, \dots, z_k, y_1, \dots, y_t))^{(s+1)}$ does not have tuples with the first k elements from B. If $t = 0$ and $k = 2$ then we get a contradiction with Corollary 8.12.1. If $t + k \geqslant 3$ then we get a contradiction with Corollary 7.11.2. □

Theorem 9.4. *Suppose $D^{(\top)}$ is a 1-consistent PC reduction for a cycle-consistent irreducible CSP instance Θ, Θ is not linked and not fragmented. Then there exists a minimal strategy $D^{(0)}, D^{(1)}, \dots, D^{(s)}$ for Θ such that the solution set of $\Theta^{(1)}$ is subdirect, the reductions $D^{(2)}, \dots, D^{(s)}$ are nonlinear, $D_x^{(s)} \subseteq D_x^{(\top)}$ for every variable x.*

Proof. Since Θ is not linked, there exists a maximal congruence σ_x on D_x for a variable x of Θ such that $\mathrm{LinkedCon}(\Theta, x) \subseteq \sigma_x$. Choose an equivalence class $D_x^{(1)}$ of σ_x with a nonempty intersection with $D_x^{(\top)}$. For every variable y by $\Theta_y^{(1)}$ we denote the set of all elements of D_y linked to an element of $D_x^{(1)}$. Since Θ is irreducible, the solution set of $\Theta^{(1)}$ is subdirect. If D_x/σ_x is a PC algebra, then $D^{(1)}$ is a PC reduction, otherwise, $D^{(1)}$ is a linear reduction.

We build the remaining part of the strategy in the following way. Suppose we already have $D^{(0)}, D^{(1)}, \dots, D^{(t)}$, where the reductions $D^{(2)}, \dots, D^{(t)}$ are absorbing or central. If there exists a binary absorption or a center on $D_y^{(t)}$ for some y, then by Theorems 9.2, 9.3 we can find the next minimal 1-consistent absorbing or central reduction $D^{(t+1)}$.

Suppose there is no a binary absorption and a center on $D_y^{(t)}$ for every y. Put $D_y^{(\perp)} = D_y^{(\top)} \cap D_y^{(t)}$ for every variable y. First, let us show that $D_y^{(\perp)}$ is a PC subuniverse of $D_y^{(t)}$ for every variable y. Consider a maximal congruence σ on D_y such that D_y/σ is a PC algebra. If $D^{(1)}$ is a PC reduction, then Lemma 7.11 implies that either $\sigma^{(1)}$ has just one equivalence class, or $D_y^{(1)}/\sigma^{(1)}$ is isomorphic to D_y/σ. Since the reductions $D^{(2)}, \dots, D^{(t)}$ are absorbing or central and $D_y^{(1)}/\sigma^{(1)}$ has no binary absorption and center, $D_y^{(i+1)}/\sigma^{(i+1)}$ is isomorphic to $D_y^{(i)}/\sigma^{(i)}$ for every $i \in \{1, \dots, t-1\}$. Thus, we can prove that either $\sigma^{(t)}$ has just one equivalence class, or $D_y^{(t)}/\sigma^{(t)}$ is a PC algebra, which means that $D^{(\perp)}$ is a PC reduction.

Then we apply Lemma 9.1 to find a 1-consistent reduction smaller than $D^{(\perp)}$. If we cannot find it, then there exists a tree-formula Υ such that $\Upsilon^{(\perp)}$ has no solutions. Choose a minimal set of variables $y_1, \dots, y_k$ from Υ such that $(\Upsilon^{(t)}(y_1, \dots, y_k))^{(\perp)}$ is empty. If $k \geqslant 3$ then we get a contradiction with Corollary 7.11.2.

Suppose $k = 2$. Let F_i be the subuniverse of D_{y_2} defined by $\exists y_1 \Upsilon^{(i)}(y_1, y_2) \wedge y_1 \in D_{y_1}^{(\top)}$. Then there exists a maximal PC congruence δ on D_{y_2} and an equivalence class E of δ such that $D_{y_2}^{(\top)} \subseteq E$ and $F_t \cap E = \varnothing$. It is easy to check that $F_1/\delta^{(1)}$ contains more than one element. Since the reductions $D^{(2)}, \dots, D^{(t)}$ are absorbing or central and D_{y_2}/δ has no binary absorption and center, $F_{i+1}/\delta^{(i+1)}$ is isomorphic to $F_i/\delta^{(i)}$ for every $i \in \{0, 1, \dots, t-1\}$. This contradicts the fact that $F_t \cap E = \varnothing$.

This contradiction proves that there exists a 1-consistent reduction $D^{(\triangle)}$ smaller than $D^{(\perp)}$ such that for every variable y the new domain $D_y^{(\triangle)}$ can be defined by a tree-formula Υ_y. By Corollary 7.11.1, for every y the domain $D_y^{(\triangle)}$ is a PC subuniverse. It remains to apply Theorem 9.3 to find a minimal reduction $D^{(t+1)}$ smaller than $D_y^{(\triangle)}$, put $s = t+1$, and finish the strategy. □

9.2 Existence of a linked connected component

In this subsection we prove that all constraints in a crucial instance have the parallelogram property, show that we can always find a linked connected component with required properties, prove that we cannot loose the only solution while applying a minimal nonlinear reduction.

Theorem 9.5. *Suppose $D^{(0)}, \dots, D^{(s)}$ is a minimal strategy for a cycle-consistent irreducible CSP instance Θ, the constraint $\rho(x_1, \dots, x_n)$ is crucial in $D^{(s)}$. Then ρ is a critical relation with the parallelogram property.*

Theorem 9.6. *Suppose $D^{(0)}, \dots, D^{(s)}$ is a minimal strategy for a cycle-consistent irreducible CSP instance Θ, $\Upsilon(x_1, \dots, x_n)$ is a subconstraint of Θ, the solution set of $\Upsilon^{(s)}$ is subdirect,* $\mathrm{Var}(\Upsilon) = \{x_1, \dots, x_n, u_1, \dots, u_t\}$,

$$\Omega = \Upsilon_{x_1,\dots,x_k,u_1,\dots,u_t}^{y_1,\dots,y_k,v_1,\dots,v_t} \wedge \Upsilon_{x_{k+1},\dots,x_n,u_1,\dots,u_t}^{y_{k+1},\dots,y_n,v_{t+1},\dots,v_{2t}} \wedge \Upsilon_{x_1,\dots,x_n,u_1,\dots,u_t}^{y_1,\dots,y_n,v_{2t+1},\dots,v_{3t}},$$

the domains of the variables x_j, y_j are the same for every $j \in \{1, \dots, n\}$, the domains of the variables $u_i, v_i, v_{t+i}, v_{2t+i}$ are the same for every $i \in \{1, \dots, t\}$, $\Theta^{(s)}$ has no solutions. Then $(\Theta \setminus \Upsilon) \cup \Omega$ has no solutions in $D^{(s)}$.

Theorem 9.7. *Suppose $D^{(0)}, D^{(1)}$ is a minimal strategy for a cycle-consistent irreducible CSP instance Θ, $\Omega^{(1)}(x_1, \dots, x_n)$ is a subconstraint of $\Theta^{(1)}$, the solution set of $\Omega^{(1)}$ is subdirect, $\Theta \setminus \Omega$ has a solution in $D^{(1)}$, Θ has no solutions in $D^{(1)}$. Then there exist formulas $\Omega_1, \dots, \Omega_t \in$ Coverings(Ω) such that $(\Theta \setminus \Omega) \cup \Omega_1 \cup \dots \cup \Omega_t$ has no solutions in $D^{(1)}$ and $\Omega_i^{(1)}(x_1, \dots, x_n)$ defines a subdirect key relation with the parallelogram property for every i.*

Theorem 9.8. *Suppose $D^{(1)}$ is a proper minimal 1-consistent reduction of a cycle-consistent irreducible CSP instance Θ, Θ is crucial in $D^{(1)}$ and not connected. Then there exists an instance $\Theta' \in$ ExpCov(Θ) that is crucial in $D^{(1)}$ and contains a linked connected component whose solution set is not subdirect.*

Theorem 9.9. *Suppose $D^{(1)}$ is a minimal 1-consistent nonlinear reduction of a cycle-consistent irreducible CSP instance Θ. If Θ has a solution then it has a solution in $D^{(1)}$.*

To prove these theorems we need to introduce a partial order on on the reductions (domain sets). Suppose we have two domain sets $D^{(\top)}$ and $D^{(\perp)}$. We say that $D^{(\perp)} \leqslant D^{(\top)}$ if for every $D_y^{(\perp)}$ one of the following conditions hold

1. there exists a variable x such that $D_y^{(\perp)} = D_x^{(\top)}$.
2. there exists a variable x such that $D_y^{(\perp)} \subsetneq D_x^{(\top)}$; there does not exist a variable z such that $D_z^{(\perp)} = D_x^{(\top)}$.

We say that $D^{(\perp)} < D^{(\top)}$ if $D^{(\perp)} \leqslant D^{(\top)}$ and $D^{(\top)} \nleqslant D^{(\perp)}$. It is not hard to see that the relation $\leqslant$ is transitive and there does not exist an infinite descending chain of reductions.

We prove theorems of this subsection simultaneously by the induction on the size of the reductions (domain sets). Let $D^{(\perp)}$ be a domain set. Assume that Theorems 9.7, 9.8, and 9.9 hold if $D^{(1)} < D^{(\perp)}$, and Theorems 9.5 and 9.6 hold if $D^{(s)} < D^{(\perp)}$. Let us prove Theorems 9.7, 9.8, and 9.9 for $D^{(1)} = D^{(\perp)}$, and Theorems 9.5 and 9.6 for $D^{(s)} = D^{(\perp)}$.

Theorem 9.5. *Suppose $D^{(0)}, \dots, D^{(s)}$ is a minimal strategy for a cycle-consistent irreducible CSP instance Θ, the constraint $\rho(x_1, \dots, x_n)$ is crucial in $D^{(s)}$. Then ρ is a critical relation with the parallelogram property.*

Proof. Since $\rho(x_1, \dots, x_n)$ is crucial, ρ is a critical relation. Let Θ' be obtained from Θ by replacement of $\rho(x_1, \dots, x_n)$ by all weaker constraints. By Lemma 6.1, Θ' is cycle-consistent and irreducible.

Assume that $|D_x^{(s)}| = 1$ for every variable x. Since the reduction $D^{(s)}$ is 1-consistent, we get a solution, which contradicts the fact that Θ has no solutions in $D^{(s)}$.

If we have a binary absorption, or a center, or a proper PC subuniverse on some domain $D_x^{(s)}$, then by Theorems 9.2, 9.3, there exists a minimal nonlinear reduction $D^{(s+1)}$ for Θ. By Lemma 8.2, $\Theta'^{(s)}$ is cycle-consistent and irreducible. Hence, by Theorem 9.9 Θ' has a solution in $D^{(s+1)}$. Hence, $\rho(x_1, \dots, x_n)$ is crucial in $D^{(s+1)}$. By the inductive assumption ρ has the parallelogram property.

It remains to consider the case when ConLin$(D_x^{(s)})$ is proper for every x such that $|D_x^{(s)}| > 1$. Let α be a solution of Θ' in $D^{(s)}$. Let the projection of α onto the variables $x_1, \dots, x_n$ be $(a_1, \dots, a_n)$.

Assume that ρ does not have the parallelogram property. Without loss of generality we can assume that there exist $c_1, \dots, c_n$ and $d_1, \dots, d_n$ such that

$$\begin{aligned}(c_1, \dots, c_k, c_{k+1}, \dots, c_n) \notin \rho,\\ (c_1, \dots, c_k, d_{k+1}, \dots, d_n) \in \rho,\\ (d_1, \dots, d_k, c_{k+1}, \dots, c_n) \in \rho,\\ (d_1, \dots, d_k, d_{k+1}, \dots, d_n) \in \rho.\end{aligned}$$

Put

$$\begin{aligned}\rho'(x_1, \dots, x_n) = \exists y_1 \dots \exists y_n\ \rho(x_1, \dots, x_k, y_{k+1}, \dots, y_n) \wedge \\ \rho(y_1, \dots, y_k, x_{k+1}, \dots, x_n) \wedge \rho(y_1, \dots, y_k, y_{k+1}, \dots, y_n).\end{aligned}$$

Obviously, $\rho \subsetneq \rho'$ and $\rho' \in \Gamma$, therefore $(a_1, \dots, a_n) \in \rho'$. Hence, there exist $b_1, \dots, b_n$ such that

$$\begin{aligned}(a_1, \dots, a_k, b_{k+1}, \dots, b_n) \in \rho,\\ (b_1, \dots, b_k, a_{k+1}, \dots, a_n) \in \rho,\\ (b_1, \dots, b_k, b_{k+1}, \dots, b_n) \in \rho.\end{aligned}$$

By Lemma 8.13, there exists a tuple $(e_1, \dots, e_n) \in \rho$ such that $(a_i, e_i) \in \mathrm{ConLin}(D_{x_i}^{(s)})$ for every i. It is easy to see that $\Theta^{(s)}$ factorized by $\mathrm{ConLin}(D_x^{(s)})$ for every x has a solution corresponding to α. By Lemma 7.15.1, the minimal linear reduction corresponding to this solution is 1-consistent. We denote this reduction by $D^{(s+1)}$. Since Θ' has a solution in $D^{(s+1)}$, $\rho(x_1, \dots, x_n)$ is crucial in $D^{(s+1)}$. We get a longer minimal strategy with smaller $D^{(s+1)}$, hence by the inductive assumption the relation ρ is a critical relation with the parallelogram property. □

Theorem 9.6. *Suppose $D^{(0)}, \dots, D^{(s)}$ is a minimal strategy for a cycle-consistent irreducible CSP instance Θ, $\Upsilon(x_1, \dots, x_n)$ is a subconstraint of Θ, the solution set of $\Upsilon^{(s)}$ is subdirect,* $\mathrm{Var}(\Upsilon) = \{x_1, \dots, x_n, u_1, \dots, u_t\}$,

$$\Omega = \Upsilon_{x_1,\dots,x_k,u_1,\dots,u_t}^{y_1,\dots,y_k,v_1,\dots,v_t} \wedge \Upsilon_{x_{k+1},\dots,x_n,u_1,\dots,u_t}^{y_{k+1},\dots,y_n,v_{t+1},\dots,v_{2t}} \wedge \Upsilon_{x_1,\dots,x_n,u_1,\dots,u_t}^{y_1,\dots,y_n,v_{2t+1},\dots,v_{3t}},$$

the domains of the variables x_j, y_j are the same for every $j \in \{1, \dots, n\}$, the domains of the variables $u_i, v_i, v_{t+i}, v_{2t+i}$ are the same for every $i \in \{1, \dots, t\}$, $\Theta^{(s)}$ has no solutions. Then $(\Theta \setminus \Upsilon) \cup \Omega$ has no solutions in $D^{(s)}$.

Proof. Put $\Theta' = (\Theta \setminus \Upsilon) \cup \Omega$. Assume that Θ' has a solution in $D^{(s)}$. Then we build a sequence of reductions $D^{(s)}, D^{(s+1)}, \dots, D^{(q)}$, which is a strategy for $\Upsilon^{(s)}$ and a minimal strategy for $((\Theta \setminus \Upsilon) \cup \Omega)^{(s)}$. Also we want Θ' to have a solution in $D^{(q)}$, and the solution set of $\Upsilon^{(j)}$ to be subdirect for every $j \in \{s, \dots, q\}$.

We will prove that we can make this sequence longer while $|D_{x_i}^{(q)}| > 1$ for some i. Assume that $|D_{x_i}^{(q)}| = 1$ for every i. Since $D^{(s)}, D^{(s+1)}, \dots, D^{(q)}$ is a strategy for Θ, Θ has a solution in $D^{(q)}$, which contradicts the fact that Θ has no solutions in $D^{(s)}$.

If we have a binary absorption, or a center, or a proper PC congruence on some domain $D_x^{(q)}$, then by Theorems 9.2, 9.3 there exists a minimal 1-consistent nonlinear reduction $D^{(q+1)}$ for $\Theta \cup \Omega$. By Lemma 8.2, $\Theta'^{(q)}$ is cycle-consistent and irreducible. By Theorem 9.9 Θ' has a solution in $D^{(q+1)}$ and Υ has a solution in $D^{(q+1)}$. By Lemma 8.1, the solution set of $\Upsilon^{(q+1)}$ is subdirect. Thus, we made the sequence longer.

It remains to consider the case when $\text{ConLin}(D_x^{(q)})$ is proper for every x such that $|D_x^{(q)}| > 1$. Let α be a solution of Θ' in $D^{(s)}$. For all variables x but $u_1, \dots, u_t$, let $D_x^{(q+1)}$ be equal to the equivalence class of $\text{ConLin}(D_x^{(q)})$ corresponding to the solution α. Let the projection of α onto the variables $x_1, \dots, x_n$ be $(a_1, \dots, a_n)$. Suppose $\Upsilon^{(s)}(x_1, \dots, x_n)$ defines the relation ρ. Since α is a solution of $\Theta'^{(s)}$, there exist $b_1, \dots, b_n$ such that

$$(a_1, \dots, a_k, b_{k+1}, \dots, b_n) \in \rho,$$
$$(b_1, \dots, b_k, a_{k+1}, \dots, a_n) \in \rho,$$
$$(b_1, \dots, b_k, b_{k+1}, \dots, b_n) \in \rho.$$

By Lemma 8.13, there exists a tuple $(d_1, \dots, d_n) \in \rho$ such that $(a_i, d_i) \in \text{ConLin}(D_{x_i}^{(q)})$ for every i. Therefore, $(\Upsilon^{(s)}(x_1, \dots, x_n))^{(q+1)}$ is not empty. Let us show that $(\Upsilon^{(q)}(x_1, \dots, x_n))^{(q+1)}$ is not empty. Assume the opposite. Let $j \geqslant s$ be the minimal number such that the relation defined by $(\Upsilon^{(j+1)}(x_1, \dots, x_n))^{(q+1)}$ is empty. If the reduction $D^{(j+1)}$ is not linear, we get a contradiction with Theorem 8.12. If the reduction $D^{(j+1)}$ is linear then it follows from the construction (see below) that $(\Upsilon^{(j+1)}(x_1, \dots, x_n))^{(q+1)}$ is not empty. Thus, we can prove that $(\Upsilon^{(q)}(x_1, \dots, x_n))^{(q+1)}$ is not empty.

Let ρ' be obtained from $\Upsilon(x_1, \dots, x_n, u_1, \dots, u_t)$ by restricting the variables $x_1, \dots, x_n$ to $D^{(q+1)}$. Let $D_{u_i}^{(q+1)}$ be the projection of ρ' onto u_i. By Corollary 7.15.1, the reduction $D^{(q+1)}$ is a 1-consistent linear reduction. Thus, we get a longer strategy such that $\Theta'^{(q+1)}$ has a solution. □

Theorem 9.7. *Suppose $D^{(0)}, D^{(1)}$ is a minimal strategy for a cycle-consistent irreducible CSP instance Θ, $\Omega^{(1)}(x_1, \dots, x_n)$ is a subconstraint of $\Theta^{(1)}$, the solution set of $\Omega^{(1)}$ is subdirect, $\Theta \setminus \Omega$ has a solution in $D^{(1)}$, Θ has no solutions in $D^{(1)}$. Then there exist formulas $\Omega_1, \dots, \Omega_t \in$* Coverings$(\Omega)$ *such that $(\Theta \setminus \Omega) \cup \Omega_1 \cup \dots \cup \Omega_t$ has no solutions in $D^{(1)}$ and $\Omega_i^{(1)}(x_1, \dots, x_n)$ defines a subdirect key relation with the parallelogram property for every i.*

Proof. Let Σ be the set of all constraints defined by $\Upsilon^{(1)}(x_1, \dots, x_n)$ where $\Upsilon \in \text{Coverings}(\Omega)$. It is easy to see that we can find $\Sigma_0 \subseteq \Sigma$ such that the instance $(\Theta^{(1)} \setminus \Omega^{(1)}) \cup \Sigma_0$ has no solutions, but if we replace any constraint of Σ_0 by all weaker constraints from Σ then we get an instance with a solution.

Let $\Sigma_0 = \{C_1, \dots, C_t\}$. It is easy to see that for every i we can find a tuple α_i such that C_i is maximal without α_i in Σ. Otherwise, we take a maximal constraint without α in Σ for every $\alpha \notin C_i$, and replace C_i by all such constraints. Obviously, the instance does not get a solution after the replacement.

By Corollary 8.5.1, C_i is a key constraint for every i. Therefore we get a sequence of formulas $\Omega_1, \dots, \Omega_t \in \text{Coverings}(\Omega)$ that define constraints $C_1, \dots, C_t$ in $D^{(1)}$. We choose variables in the formulas so that the only common variables of $\Omega_1, \dots, \Omega_t$ are $x_1, \dots, x_n$. It follows from Theorem 9.6 that C_i has the parallelogram property for every i. □

To prove the next theorem we define several transformations of a CSP instance Θ.

Transformation $T_1(\Theta)$: make the instance crucial in $D^{(1)}$. Using Remark 1, we replace constraints by all weaker constraints until we get a CSP instance that is crucial in $D^{(1)}$. After that we remove all isolated variables, that is, the variables that do not appear in any constraint.

Below we assume that the instance Θ is crucial in $D^{(1)}$, which by Theorem 9.5 means that every constraint in Θ has the parallelogram property.

Transformation $T_2(\Theta, C_1, C_2, x)$: split two constraints with a common variable. Assume that two constraints C_1 and C_2 have common variable x. Let Ω_i be the set of all constraints $C \in \Theta$ such that $\mathrm{Con}(C, x) = \mathrm{Con}(C_i, x)$ for $i \in \{1, 2\}$. Let Ω_0 be the set of all constraints $C \in \Theta \setminus (\Omega_1 \cup \Omega_2)$ containing x. Put $\sigma_i = \mathrm{Con}(C_i, x)$ for $i \in \{1, 2\}$. We transform our instance in the following way.

1. We remove all constraints with x.
2. Choose 2 new variables x_1 and x_2.
3. Add all constraints from Ω_0 twice (with x_1 and x_2 instead of x).
4. Add all constraints from Ω_1 with x_1 instead of x.
5. Add all constraints from Ω_2 with x_2 instead of x.
6. Add the constraints $\sigma_1^*(x_1, x_2)$ and $\sigma_2^*(x_1, x_2)$.

Lemma 9.10. *Suppose $D^{(1)}$ is a proper minimal 1-consistent reduction of a cycle-consistent irreducible CSP instance Θ, Θ is crucial in $D^{(1)}$, C_1 and C_2 are constraints of Θ with a common variable x, C_1 and C_2 are not adjacent in x. Then the instance $T_2(\Theta, C_1, C_2, x)$ has no solutions in $D^{(1)}$.*

Proof. Let $\Theta' = T_2(\Theta, C_1, C_2, x)$, σ be the intersection of all congruences from $\mathrm{Con}(\Omega_0, x)$.

Assume that Θ' has a solution in $D^{(1)}$. Suppose $(x_1, x_2) = (a_1, a_2)$ in this solution. Put $\Upsilon = \sigma_1(x_1, x) \wedge \sigma_2(x_2, x) \wedge \sigma(x_2, x)$. It is not hard to see that $\Theta' \wedge \Upsilon$ has no solutions in $D^{(1)}$ (otherwise we could take this solution as a solution to $\Theta^{(1)}$). We apply Theorem 9.7 to the subconstraint $\Upsilon(x_1, x_2)$. Then $\Upsilon(x_1, x_2)$ can be replaced by a sequence of formulas $\Omega_1, \ldots, \Omega_t \in \mathrm{Coverings}(\Omega)$. Assume that $\Omega_i^{(1)}(x_1, x_2)$ defines a relation ρ_i. It is easy to see that every ρ_i is a reflexive relation with the parallelogram property, that is a congruence on $D_{x_1}^{(1)}$. If the reduction $D^{(1)}$ is nonlinear then by ω_i we denote the relation defined by $\Omega_i(x_1, x_2)$. If the reduction $D^{(1)}$ is linear then by ω_i we denote the relation defined by $\Omega_i'(x_1, x_2, u_1)$ from Lemma 8.7. We know from Lemma 8.7 and Lemma 8.6 that $\mathrm{Con}(\omega_i, 1)^{(1)} = \mathrm{Con}(\rho_i, 1)$ and $\mathrm{Con}(\omega_i, 2)^{(1)} = \mathrm{Con}(\rho_i, 2)$.

Case 1. Assume that $\rho_i \neq \sigma_1^{(1)}$ for every i, then $\mathrm{Con}(\omega_i, 1) \supseteq \sigma_1{}^*$. Hence $\rho_i \supseteq (\sigma_1{}^*)^{(1)}$ for every i. Then we may put $x_1 = a_1$ and $x = x_2 = a_2$ to get a solution for $\Theta' \wedge \Upsilon$ in $D^{(1)}$, which contradicts our assumption.

Case 2. Assume that $\rho_i = \sigma_1^{(1)}$ for some i. Since $(a_1, a_2) \in (\sigma_1{}^*)^{(1)} \setminus \sigma_1$ and $\mathrm{Con}(\omega_i, 1)^{(1)} = \mathrm{Con}(\rho_i, 1)$, we have $\mathrm{Con}(\omega_i, 1) \not\supseteq \sigma_1{}^*$. Hence $\mathrm{Con}(\omega_i, 1) = \sigma_1$. Suppose $D^{(1)}$ is a nonlinear reduction. $\Upsilon(x_1, x_2)$ contains $\sigma_2 \cap \sigma$, and therefore $\sigma_2 \cap \sigma \subseteq \mathrm{Con}(\omega_i, 1) = \sigma_1$. In the same way we can show that $\sigma_1 \cap \sigma \subseteq \sigma_2$. Since, $(a_1, a_2) \in \sigma \setminus \sigma_1$, by Lemma 8.4 C_1 and C_2 are adjacent in x, which contradicts our assumptions. Similarly, if $D^{(1)}$ is a linear reduction, $\sigma_2 \cap \sigma \cap \mathrm{ConLin}(D_x) \subseteq \mathrm{Con}(\omega_i, 1) = \sigma_1$, $\sigma_1 \cap \sigma \cap \mathrm{ConLin}(D_x) \subseteq \sigma_2$, $(a_1, a_2) \in (\sigma \cap \mathrm{ConLin}(D_x)) \setminus \sigma_1$, and Lemma 8.4 gives a contradiction. □

For a $\Omega \subseteq \Theta$ by $\mathrm{MinVar}(\Omega, \Theta)$ we denote the set of all variables x such that $\mathrm{Con}(C, x)$ is minimal in $\mathrm{Con}(\Theta, x)$ for some $C \in \Omega$.

Transformation $T_3(\Theta, \Omega)$ for a connected component Ω. Let $\mathrm{MinVar}(\Omega, \Theta) = \{x_1, \ldots, x_s\}$. Let us define the new instance in the following way.

1. Choose new variables $x_1', \ldots, x_s'$.
2. Replace the variables $x_1, \ldots, x_s$ in $\Theta \setminus \Omega$ by $x_1', \ldots, x_s'$.
3. Add a copy of Ω with all the variables $x_1, \ldots, x_s$ replaced by $x_1', \ldots, x_s'$ and all constraints replaced by their covers.

Lemma 9.11. *Suppose $D^{(1)}$ is a proper minimal 1-consistent reduction of a cycle-consistent irreducible CSP instance Θ, Θ is crucial in $D^{(1)}$, Ω is a connected component of Θ, $\text{LinkedCon}(\Omega, y) = \text{Con}(C, y)$ for every variable y and every constraint $C \in \Omega$ with y, $\text{MinVar}(\Omega, \Theta) \neq \text{Var}(\Omega)$. Then the instance $T_3(\Theta, \Omega)$ has no solutions in $D^{(1)}$.*

Proof. By Ω' we denote the copy of Ω we introduce in item 3 of the definition. Since all constraints of Ω are rectangular and critical, they are also essential. Therefore, if the arity of the constraint C is greater than 2, then by Lemma 8.14, $\text{Con}(C, y) \subsetneq \text{LinkedCon}(\Omega, y)$, which means that every constraint in Ω is binary. We can imagine that we factorize all the variables x_i by the only congruence in $\text{Con}(\Omega, x_i)$. In this case, every constraint relation from Ω can be viewed as the equality relation and the cover of every constraint relation becomes the minimal binary relation containing the equality relation. It is not hard to see that any congruence in $\text{Con}(\Theta \setminus \Omega, x_i)$ factorized by the only congruence in $\text{Con}(\Omega, x_i)$ contains this binary relation, which means that every congruence in $\text{Con}(\Theta \setminus \Omega, x_i)$ contains $\text{LinkedCon}(\Omega', x_i')$.

Assume that $T_3(\Theta, \Omega)$ has a solution in $D^{(1)}$ with

$$(x_1, \ldots, x_s, x_1', \ldots, x_s') = (b_1, \ldots, b_s, b_1', \ldots, b_s').$$

Since $(b_i, b_i') \in \text{LinkedCon}(\Omega', x_i')$ for every i, we can assign

$$(x_1, \ldots, x_s, x_1', \ldots, x_s') = (b_1, \ldots, b_s, b_1, \ldots, b_s).$$

to get a solution of $\Theta^{(1)}$ (the remaining variables take on the same values). This contradiction proves that $T_3(\Theta, \Omega)$ has no solutions in $D^{(1)}$. □

Transformation $T_4(\Theta, \Omega, u)$ for a connected component Ω and a variable u. Let $\text{MinVar}(\Omega, \Theta) = \{x_1, \ldots, x_s\}$. Let us define the new instance in the following way.

1. Choose new variables $x_1', \ldots, x_s'$.
2. Rename the variables $x_1, \ldots, x_s$ by $x_1', \ldots, x_s'$ in $\Theta \setminus \Omega$.
3. Add the covers of all constraints from Ω with $x_1', \ldots, x_s'$ instead of $x_1, \ldots, x_s$.
4. For every i and every $\sigma \in \text{Con}(\Theta \setminus \Omega, x_i)$ add the constraint $\sigma^*(x_i, x_i')$.
5. If $u = x_h$, then add the constraint $\delta_h(x_h, x_h')$, where $\zeta \in \text{Con}(\Omega, x_h)$ and $\delta_h = \text{Opt}(\zeta)$.

Note that we allow to put $\varnothing$ instead of u.

Lemma 9.12. *Suppose $D^{(1)}$ is a proper minimal 1-consistent reduction of a cycle-consistent irreducible CSP instance Θ, Θ is crucial in $D^{(1)}$, Ω is a connected component of Θ, the solution set of Ω is subdirect, Ω has a solution in $D^{(1)}$, $\text{LinkedCon}(\Omega, y) \neq \text{Con}(C, y)$ for some variable y and some constraint $C \in \Omega$, $u \in \text{MinVar}(\Omega, \Theta)$ or $u = \varnothing$ and $\text{MinVar}(\Omega, \Theta) \neq \text{Var}(\Omega)$. Then the instance $T_4(\Theta, \Omega, u)$ has no solutions in $D^{(1)}$.*

Proof. To prove this lemma we consider a different transformation defined as follows

1. Choose new variables $x_1', \ldots, x_s'$ and $x_1'', \ldots, x_s''$.
2. Add a copy of Ω to Θ with all the variables $x_1, \ldots, x_s$ replaced by $x_1', \ldots, x_s'$. The copy we denote by Ω'.
3. Rename $x_1, \ldots, x_s$ in $\Theta \setminus \Omega$ by $x_1'', \ldots, x_s''$.

4. For every i and every $\sigma \in \mathrm{Con}(\Theta \setminus \Omega, x_i)$ add a new variable y and add the constraints $\sigma(x_i', y)$ and $\sigma(x_i'', y)$.

5. For every i and every $\sigma \in \mathrm{Con}(\Theta \setminus \Omega, x_i)$ add the constraint $\sigma^*(x_i, x_i'')$.

6. If $u = x_h$, then add the constraint $\delta_h(x_h, x_h')$, where $\zeta \in \mathrm{Con}(\Omega, x_h)$ and $\delta_h = \mathrm{Opt}(\zeta)$.

It is easy to see that the obtained instance has no solutions in $D^{(1)}$. Then we replace constraints from Ω' containing at least one of the variables $x_1', \ldots, x_s'$ by their covers step by step. Thus, in one step we replace just one constraint from Ω'. We consider two cases.

Assume that after all replacements we get an instance without solutions in $D^{(1)}$. It is not hard to see that any solution of $T_4(\Theta, \Omega, u)$ gives a solution of this instance: if $x_i' = a_i'$ in the solution of $T_4(\Theta, \Omega, u)$, then we put $x_i' = x_i'' = y = a_i$ in this instance for every i and the corresponding y's. This completes this case.

Assume that after some replacement the instance gets a solution in $D^{(1)}$. Suppose the instance before this replacement is Θ' and the corresponding constraint to be replaced is C. Choose a variable $x_l' \in \mathrm{Var}(C)$.

Let $\delta = \mathrm{Con}(C, x_l')$, ρ be an optimal bridge from δ to δ. Let us define a new bridge by

$$\rho'(u_1, u_2, u_3, u_4) = \exists v_1 \exists v_2 \rho(u_1, u_2, v_1, v_2) \wedge \rho(u_3, u_4, v_1, v_2) \wedge \rho(u_1, u_1, u_3, u_3) \wedge \delta^*(u_3, u_4).$$

It is not hard to see that ρ' is still an optimal bridge.

Then we change Θ' in the following way. We add three new variables u_1, u_2, x_l''', replace x_l' in C by x_l''', add the constraint $\rho'(x_l', x_l''', u_1, u_2)$ and the constraint $\delta(u_1, u_2)$. The new instance we denote by Θ''. Obviously, Θ'' has no solutions in $D^{(1)}$.

By Υ we denote all constraints of Θ'' containing x_j' for some j or x_l'''. Suppose $\mathrm{Var}(\Omega) \setminus \mathrm{MinVar}(\Omega, \Theta) = \{z_1, \ldots, z_n\}$. Let $\{y_1, \ldots, y_t\}$ be the set of all variables of Υ except for $z_1, \ldots, z_n$, $x_1', \ldots, x_s'$, x_h, u_1, u_2, and x_l'''. Suppose that the variable x_{i_j} is the corresponding variable and σ_j is the corresponding congruence for y_j (see Step 4 of the transformation).

Consider a subconstraint $\Upsilon(y_1, \ldots, y_t, x_h, z_1, \ldots, z_n, u_1, u_2)$. Since the solution set of Ω is subdirect, by Lemma 8.1 we know that the solution set of $\Upsilon^{(1)}$ is subdirect. Then by Theorem 9.7, we can find $\Upsilon_1, \ldots, \Upsilon_v \in \mathrm{Coverings}(\Upsilon)$ such that $\Upsilon_i^{(1)}(y_1, \ldots, y_t, x_h, z_1, \ldots, z_n, u_1, u_2)$ defines a key relation ρ_i with the parallelogram property for every i.

Let us define a relation ω_i for every $i \in \{1, \ldots, v\}$. If $D^{(1)}$ is a nonlinear reduction, it is the relation defined by $\Upsilon_i(y_1, \ldots, y_t, x_h, z_1, \ldots, z_n, u_1, u_2)$. If $D^{(1)}$ is a linear reduction, it is the relation defined by $\Upsilon_i'(y_1, \ldots, y_t, x_h, z_1, \ldots, z_n, u_1, u_2, q_1, \ldots, q_r)$, where Υ_i' is the formula from Lemma 8.7. We know from Lemmas 8.7 and 8.6 that $\mathrm{Con}(\omega_i, j)^{(1)} = \mathrm{Con}(\rho_i, j)$ for every $j \in \{1, 2, \ldots, t+n+3\}$.

We know that if we remove the constraint $\delta(u_1, u_2)$ from Θ'', then we get a solution in $D^{(1)}$. Let

$$(x_1, \ldots, x_s, x_1', \ldots, x_s', x_1'', \ldots, x_s'', y_1, \ldots, y_t, z_1, \ldots, z_n, u_1, u_2) = \\ (a_1, \ldots, a_s, a_1', \ldots, a_s', a_1'', \ldots, a_s'', d_1, \ldots, d_t, b_1, \ldots, b_n, c_1, c_2)$$

in this solution. Choose k such that ρ_k omits the tuple $(d_1, \ldots, d_t, a_h, b_1, \ldots, b_n, c_1, c_1)$. For every j we put $d_j' = a_{i_j}$. It is easy to see that $(d_1', \ldots, d_t', a_h, b_1, \ldots, b_n, a_l, a_l) \in \rho_k$.

We want to show that $(a_l, a_l, c_1, c_1) \in \rho'$. We consider two cases.

Case 1. Suppose $u = \varnothing$ and $n > 0$. In this case a_l and a_l' are linked in Ω'', where Ω'' is the instance obtained from Ω by replacing every constraint by its cover. We apply Theorem 8.15 to get a bridge from δ to δ containing (a_l, a_l, a_l', a_l'). Then we compose this bridge with the bridge ρ' to obtain a bridge from δ to δ containing (a_l, a_l, c_1, c_1). Since the bridge ρ' is optimal, we have $(a_l, a_l, c_1, c_1) \in \rho'$.

Case 2. Suppose $u = x_h$. We know that a_l and a_h are linked in Ω, a'_h and a'_l are linked in Ω'', where Ω'' is the instance obtained from Ω by replacing every constraint by its cover. We apply Theorem 8.15 to get a bridge from δ to ζ containing (a_l, a_l, a_h, a_h) and a bridge from ζ to δ containing (a'_h, a'_h, a'_l, a'_l). Then we compose these two bridges with the optimal bridge from Step 6 and the bridge ρ' to obtain a bridge from δ to δ containing (a_l, a_l, c_1, c_1). Since the bridge ρ' is optimal, we have $(a_l, a_l, c_1, c_1) \in \rho'$.

Therefore, $(d'_1, \dots, d'_t, a_h, b_1, \dots, b_n, c_1, c_1) \in \rho_k$. Also, we know that ρ_k is a key relation with the parallelogram property and $(d_1, \dots, d_t, a_h, b_1, \dots, b_n, c_1, c_2) \in \rho_k$. For some j we have $(d_j, d'_j) \notin \mathrm{Con}(\rho_k, j)$, hence $(d_j, d'_j) \notin \mathrm{Con}(\omega_k, j)$. Since $(d_j, d'_j) \in ({\sigma_j}^*)^{(1)}$, the j-th variable of ω_k is rectangular and $\mathrm{Con}(\omega_k, j) = \sigma_j$. Similarly, the last variable of ρ_k is rectangular and $\mathrm{Con}(\omega_k, t+n+3) = \delta$. By Lemma 8.14 there exists a bridge ζ_1 from δ to σ_j.

Suppose $\delta_0 \in \mathrm{Con}(\Omega, x_{i_j})$. Applying Theorem 8.15 to Ω, we get a bridge ζ_2 from δ_0 to δ. Composing the bridges ζ_1 and ζ_2 we get a reflexive bridge from δ_0 to σ_j. Hence δ_0 and σ_j are adjacent. This contradicts the fact that $\sigma_j \in \mathrm{Con}(\Theta \setminus \Omega, x_{i_j})$.

□

Theorem 9.8. *Suppose $D^{(1)}$ is a proper minimal 1-consistent reduction of a cycle-consistent irreducible CSP instance Θ, Θ is crucial in $D^{(1)}$ and not connected. Then there exists an instance $\Theta' \in \mathrm{ExpCov}(\Theta)$ that is crucial in $D^{(1)}$ and contains a linked connected component whose solution set is not subdirect.*

Proof. The proof is organized as follows. Using the transformations above we build a sequence of instances $\Theta_1, \Theta_2, \dots$ such that $\Theta_{i+1} \in \mathrm{ExpCov}(\Theta_i)$. This sequence will be used to apply Theorem 8.16: variables are viewed as organisms, constraints are viewed as a relationship between variables.

First, we assign a characteristic to every variable. For a variable x of an instance Φ let Ω_1 be the set of all minimal congruences among the set $\mathrm{Con}(\Phi, x)$. Then let Ω_2 be the set of all minimal congruences among the congruences of $\mathrm{Con}(\Phi, x)$ that are not adjacent with the congruences from Ω_1. Thus, we assign a pair (Ω_1, Ω_2) to every variable x, which we denote $\xi(\Phi, x)$.

Let us introduce a partial order on the set of all characteristics. For two sets of congruences Ω_1 and Ω_2 we write $\Omega_1 \leqslant \Omega_2$ if for every $\sigma \in \Omega_1$ there exists $\delta \in \Omega_2$ such that $\delta \subseteq \sigma$. We write $\Omega_1 < \Omega_2$ if $\Omega_1 \leqslant \Omega_2$ and $\Omega_2 \nleqslant \Omega_1$.

We write $(\Omega_1, \Omega_2) \lesssim (\Omega'_1, \Omega'_2)$ if one of the following conditions hold

1. $\Omega_1 < \Omega'_1$;
2. if $\Omega_1 = \Omega'_1$ and $\Omega_2 \leqslant \Omega'_2$.
3. if $\Omega_1 = \Omega'_1$, $\Omega_2 \nleqslant \Omega'_2$, $\Omega'_2 \nleqslant \Omega_2$, $(\Omega_2 \setminus \mathrm{Opt}(\Omega_1)) < (\Omega'_2 \setminus \mathrm{Opt}(\Omega_1))$.

To use Theorem 8.16, we extend a partial order $\lesssim$ on characteristics to a linear order $\leqslant$ such that $(\Omega_1, \Omega_2) \lesssim (\Omega'_1, \Omega'_2)$ implies $(\Omega_1, \Omega_2) \leqslant (\Omega'_1, \Omega'_2)$. We say that a variable x of Θ is *weaker than* a variable x' of Θ' if $\xi(\Theta, x) < \xi(\Theta', x')$.

Second, if $\Theta' \in \mathrm{ExpCov}(\Theta)$ is connected then by Corollary 8.15.1 any two constraints of Θ' with a common variable are adjacent. Since every constraint of Θ appears in Θ', Θ is also connected. This contradiction proves that we cannot get a connected instance Θ'.

We start with $\Theta_1 = \Theta$. Suppose we already defined Θ_i.

If there exist constraints C_1 and C_2 having a common variable x such that C_1 and C_2 are not adjacent in x, $\mathrm{Con}(C_1, x)$ and $\mathrm{Con}(C_2, x)$ are minimal congruences in $\mathrm{Con}(\Theta, x)$, then put $\Theta_{i+1} = T_1(T_2(\Theta_i, C_1, C_2, x))$. By Lemma 9.10, Θ_{i+1} has no solutions in $D^{(1)}$. Note that Θ_{i+1} is always different from Θ_i.

Otherwise, we know that any two minimal congruences in $\mathrm{Con}(\Theta, x)$ for every variable x are adjacent. Since Θ_i is not connected and not fragmented, there exist a variable x and two congruences in $\mathrm{Con}(\Theta, x)$ that are not adjacent. Choose a connected component Ω containing a minimal congruence of $\mathrm{Con}(\Theta, x)$.

If Ω is not linked, then irreducibility of Θ implies that the solution set of Ω is subdirect. If Ω is linked and the solution set of Ω is not subdirect, then the theorem is proved and we stop the process. Thus, we assume that the solution set of Ω is subdirect.

Since Θ_i is crucial in $D^{(1)}$ and not connected, $\Omega^{(1)}$ has a solution. Since the solution set of Ω is subdirect, Lemma 8.1 implies that the solution set of $\Omega^{(1)}$ is also subdirect. Let $\mathrm{MinVar}(\Omega, \Theta_i) = \{x_1, \dots, x_s\}$, $\mathrm{Var}(\Omega) \setminus \mathrm{MinVar}(\Omega, \Theta_i) = \{z_1, \dots, z_n\}$.

Assume that $n = 0$ and $\mathrm{LinkedCon}(\Omega, x_j) \subseteq \sigma$ for every j and every $\sigma \in \mathrm{Con}(\Theta_i \setminus \Omega, x_j)$. If there exist a constraint in Ω and a variable z such that $\mathrm{Con}(C, z) \subsetneq \mathrm{LinkedCon}(\Omega, z)$, then we replace C by its cover. Since Θ_i is crucial in $D^{(1)}$, the new instance has a solution β in $D^{(1)}$. Let z be equal to c in β. Since the solution set of $\Omega^{(1)}$ is subdirect, there exists a solution γ of $\Omega^{(1)}$ with $z = c$. Then we build a solution of $\Theta_i^{(1)}$ with the values for x_j from γ and the values for the remaining variables from β, which gives us a contradiction.

Assume that $\mathrm{Con}(C, z) = \mathrm{LinkedCon}(\Omega, z)$ for every constraint $C \in \Omega$ and every variable z of C. Since all constraints of Ω are rectangular and critical, they are also essential. Therefore, if the arity of the constraint C is greater than 2, then by Lemma 8.14, $\mathrm{Con}(C, y) \subsetneq \mathrm{LinkedCon}(\Omega, y)$, which means that every constraint in Ω is binary. We can choose a constraint $C \in \Omega$ with a variable x_j that appears just once in Ω. Then we replace the constraint C by its cover. Since Θ_i is crucial in $D^{(1)}$, the new instance has a solution β in $D^{(1)}$. Since $\mathrm{Con}(C, x_j) \subsetneq \sigma$ for every $\sigma \in \mathrm{Con}(\Theta_i \setminus \Omega, x_j)$, we can change the value of x_j in β to get a solution of Θ_i in $D^{(1)}$ which gives us a contradiction.

Otherwise, suppose $n > 0$ and $\mathrm{LinkedCon}(\Omega, z) = \mathrm{Con}(C, z)$ for every variable z and every constraint $C \in \Omega$ with z. Then we put $\Theta_{i+1} = T_1(T_3(\Theta_i, \Omega))$. By Lemma 9.11, Θ_{i+1} has no solutions in $D^{(1)}$. Again, Θ_{i+1} is always different from Θ_i.

Otherwise, if $n > 0$, then we put $\Theta_{i+1} = T_1(T_4(\Theta_i, \Omega, \varnothing))$. If $n = 0$ and $\mathrm{LinkedCon}(\Omega, x_h) \not\subseteq \sigma$ for some h and a congruence $\sigma \in \mathrm{Con}(\Theta \setminus \Omega, x_h)$ then we put $\Theta_{i+1} = T_1(T_4(\Theta_i, \Omega, x_h))$. By Lemma 9.12, Θ_{i+1} has no solutions in $D^{(1)}$. Again, Θ_{i+1} is always different from Θ_i.

It can be checked that we always define the instance Θ_{i+1}.

It remains to explain how we build a sequence for Theorem 8.16. We consider the set of all pairs $(x, \xi(\Theta_i, x))$ as the set of organisms. Two organisms are friends if they represent the same variable, or if the corresponding variables ever appeared in one constraint. The characteristic of every variable is considered as a strength. Then the set of organisms X_i corresponds to the set of all pairs $(x, \xi(\Theta_j, x))$ for $j \leqslant i$.

It is not hard to see that any copy of any variable we generate is weaker than the original variable, which guarantees condition 1 of the theorem. Similarly, we can check that all new constraints satisfy conditions 2 and 3.

Every time we apply the transformation T_2 we replace a variable by two weaker variables. Then, if our sequence of instances is infinite, we apply the transformations T_3 and T_4 (at least one of them) infinitely many times.

A variable x is called *stable at the moment* i if all congruence in $\mathrm{Con}(\Theta_i, x)$ are adjacent. Note that if x is not stable at the moment i and a connected component $\Omega \subseteq \Theta_i$ contains a minimal congruence of $\mathrm{Con}(\Theta_i, x)$, then the transformations T_3 and T_4 make the variable x weaker.

To guarantee condition 4, we choose a connected component Ω on every step so that the following condition holds. For every variable x that is not stable at the moment i a connected component with a minimal congruence on x should be chosen at some moment $j \geqslant i$. This means that every variable will be stable at some moment. It is not hard to check that a new "friend" of a stable variable z, which may appear in the transformations T_2, T_3, or T_4, is weaker than the old "friend" of z. Moreover, if z gets a new "friend" we remove the constraint containing the old "friend" and z. Thus, every variable can have only finitely many friends.

Since Θ_i is crucial in $D^{(1)}$, it is not fragmented. We can check that Θ_i and Θ_{i+1} have at least one common variable for every i. Therefore, the set of all organisms cannot be divided into two disjoint sets. Thus, condition 5 of Theorem 8.16 cannot hold, which proves that the process will stop at some Θ_i having a linked connected component whose solution set is not subdirect. □

Theorem 9.9. *Suppose $D^{(1)}$ is a minimal 1-consistent nonlinear reduction of a cycle-consistent irreducible CSP instance Θ. If Θ has a solution then it has a solution in $D^{(1)}$.*

Proof. Assume the contrary. First, we consider the set of all minimal 1-consistent nonlinear reductions of Θ, which we denote by $\mathfrak{R}$. Then we consider an instance $\Theta' \in \mathrm{ExpCov}(\Theta)$ with the minimal positive number of reductions $D^{(\triangle)} \in \mathfrak{R}$ such that Θ' has no solutions in $D^{(\triangle)}$. Note that this transformation of Θ to Θ' can be omitted if $D^{(1)}$ is not a PC reduction. Then we weaken the instance Θ' (replace any constraint by all weaker constraints) while we still have a reduction $D^{(\triangle)} \in \mathfrak{R}$ such that Θ' has no solutions in $D^{(\triangle)}$. The obtained instance we denote by Θ''. As a result we know that for any reduction $D^{(\triangle)} \in \mathfrak{R}$ the instance Θ'' is either crucial in $D^{(\triangle)}$, or has a solution in $D^{(\triangle)}$. Choose a reduction $D^{(\triangle)}$ from $\mathfrak{R}$.

Assume that Θ'' is not linked. If $D^{(1)}$ is a PC reduction, then the statement follows from Theorem 9.4 and the inductive assumption. If $D^{(1)}$ is an absorbing or central reduction, then we choose a variable x of Θ'' and an element $c \in D_x^{(1)}$, and for every variable y by $D_y^{(\top)}$ we denote the set of all elements of D_y linked to c. Since Θ'' is irreducible, the solution set of $\Theta''^{(\top)}$ is subdirect. Therefore, $\Theta''^{(\top)}$ is irreducible and cycle-consistent. It is not hard to see that the reduction $D^{(\perp)}$, defined by $D_y^{(\perp)} = D_y^{(\top)} \cap D_y^{(\triangle)}$ for every variable y, is a 1-consistent absorbing or central reduction for $\Theta''^{(\top)}$. By the inductive assumption, $\Theta''^{(\perp)}$ has a solution, which completes this case.

Thus, we assume that Θ'' is linked. Then, by Theorem 9.5, every constraint in the obtained instance has the parallelogram property. If Θ'' is not connected, then by Theorem 9.8, there exists an instance $\Upsilon \in \mathrm{ExpCov}(\Theta'')$ that is crucial in $D^{(\triangle)}$ and contains a linked connected component Ω. If Θ'' is connected, then Θ'' is a linked connected component itself and we put $\Upsilon = \Omega = \Theta''$.

Choose a variable x appearing in a constraint $C \in \Omega$. By Lemma 8.3, $\mathrm{Con}(C, x)$ is irreducible. By Theorem 8.15.1, there exists a bridge δ from $\mathrm{Con}(C, x)$ to $\mathrm{Con}(C, x)$ such that $\delta(x, x, y, y)$ is a full relation. By Corollary 8.10.1, there exists a relation $\zeta \subseteq D_x \times D_x \times \mathbb{Z}_p$ such that $(x_1, x_2, 0) \in \zeta \Leftrightarrow (x_1, x_2) \in \mathrm{Con}(C, x)$ and $\mathrm{pr}_{1,2}(\zeta) = \mathrm{Con}(C, x)^*$. Let us replace the variable x of C in Υ by x' and add the constraint $\zeta(x, x', z)$. The obtained instance we denote by Υ'. By the assumption, Υ' has a solution with $z = 0$, and a solution in $D^{(\triangle)}$ with $z \neq 0$.

If $D^{(\triangle)}$ is an absorbing or central reduction, then by Corollaries 7.1.1, 7.6.1 the restriction of all variable of Υ' but z to $D^{(\triangle)}$ implies the corresponding restriction of the variable z. This contradicts the fact that the domain of z is $\mathbb{Z}_p$.

It remains to consider the case when $D^{(\triangle)}$ is a PC reduction. By Theorems 9.2, 9.3, for every variable y and every PC subuniverse U of D_y there exists a minimal 1-consistent PC reduction $D^{(\triangledown)} \in \mathfrak{R}$ such that $D_y^{(\triangledown)} = U$. Since Υ' has a solution in any reduction from $\mathfrak{R}$, we conclude that for every variable y and every PC subuniverse U of D_y the instance Υ' has a solution with $y \in U$. Hence, by Corollary 7.11.1, the restriction of Υ' to $D^{(\triangle)}$ implies the corresponding restriction of z, which contradicts the fact that the domain of z is $\mathbb{Z}_p$. □

9.3 Theorems from Section 4

In this subsection we assume that variables of the instance Θ are $x_1, \ldots, x_n$, and the domain of x_i is D_i for every i. The first two theorems are proved together.

Theorem 4.3. *Suppose Θ is a cycle-consistent irreducible CSP instance, B is a binary absorbing set or a center of D_i. Then Θ has a solution if and only if Θ has a solution with $x_i \in B$.*

Theorem 4.4. *Suppose Θ is a cycle-consistent irreducible CSP instance, there does not exist a binary absorbing subuniverse or a center on D_j for every j, $(D_i; w)/\sigma$ is a polynomially complete algebra, E is an equivalence class of σ. Then Θ has a solution if and only if Θ has a solution with $x_i \in E$.*

Proof. By Theorem 9.2, 9.3, there exists a smaller minimal 1-consistent reduction. By Theorem 9.9, there exists a solution in this reduction. □

Theorem 4.5. *Suppose the following conditions hold:*

1. *Θ is a linked cycle-consistent irreducible CSP instance with domain set $(D_1, \ldots, D_n)$;*
2. *there does not exist a binary absorbing subuniverse or a center on D_j for every j;*
3. *if we replace every constraint of Θ by all weaker constraints then the obtained instance has a solution with $x_i = b$ for every i and $b \in D_i$.*
4. *Θ_L is Θ factorized by the minimal linear congruences;*
5. *$(D'_1, \ldots, D'_n)$ is a solution of Θ_L, and Θ is crucial in $(D'_1, \ldots, D'_n)$.*

Then there exists a constraint $((x_{i_1}, \ldots, x_{i_s}), \rho)$ in Θ and a subuniverse ζ of $\mathbf{D_{i_1}} \times \cdots \times \mathbf{D_{i_s}} \times \mathbb{Z}_\mathbf{p}$ such that the projection of ζ onto the first s coordinates is bigger than ρ but the projection of $\zeta \cap (D_{i_1} \times \cdots \times D_{i_s} \times \{0\})$ onto the first s coordinates is equal to ρ.

Proof. Assume the contrary. We denote the reduction $(D'_1, \ldots, D'_n)$ by $D^{(1)}$. By Theorem 9.5, every constraint in Θ has the parallelogram property. If Θ is not connected, then by Theorem 9.8, there exists an instance $\Theta' \in \mathrm{ExpCov}(\Theta)$ that is crucial in $D^{(1)}$ and contains a linked connected component Ω such that the solution set of Ω is not subdirect. By condition 3), if the solution set of Ω is not subdirect then Ω contains a constraint relation from Θ. If Θ is connected, then Θ is a linked connected component itself and we put $\Omega = \Theta$. Let $((x_{i_1}, \ldots, x_{i_s}), \rho) \in \Omega$ be a constraint such that ρ is a constraint relation from Θ.

By Lemma 8.3, $\mathrm{Con}(\rho, 1)$ is an irreducible congruence. By Theorem 8.15.1, there exists a bridge δ from $\mathrm{Con}(\rho, 1)$ to $\mathrm{Con}(\rho, 1)$ such that $\delta(x, x, y, y)$ is a full relation. By Corollary 8.10.1, there exists a relation $\xi \subseteq D_{i_1} \times D_{i_1} \times \mathbb{Z}_p$ such that $(x_1, x_2, 0) \in \xi \Leftrightarrow (x_1, x_2) \in \mathrm{Con}(\rho, 1)$ and $\mathrm{pr}_{1,2}(\xi) = \mathrm{Con}(\rho, 1)^*$.

Put $\zeta(x_{i_1}, \ldots, x_{i_s}, z) = \exists x'_{i_1}\ \rho(x'_{i_1}, x_{i_2}, \ldots, x_{i_s}) \wedge \xi(x_{i_1}, x'_{i_1}, z)$. □

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