# Convergence of an Asynchronous Block-Coordinate Forward-Backward Algorithm for Convex Composite Optimization 

Cheik Traoré ${ }^{\boxed{W} *}$, Saverio Salzo ${ }^{\dagger}$ and Silvia Villa ${ }^{\text { }}$


#### Abstract

In this paper, we study the convergence properties of a randomized block-coordinate descent algorithm for the minimization of a composite convex objective function, where the blockcoordinates are updated asynchronously and randomly according to an arbitrary probability distribution. We prove that the iterates generated by the algorithm form a stochastic quasi-Fejér sequence and thus converge almost surely to a minimizer of the objective function. Moreover, we prove a general sublinear rate of convergence in expectation for the function values and a linear rate of convergence in expectation under an error bound condition of Tseng type.


Keywords. Convex optimization, asynchronous algorithms, randomized block-coordinate descent, error bounds, stochastic quasi-Fejér sequences, forward-backward algorithm, convergence rates.

AMS Mathematics Subject Classification: 65K05, 90C25, 90C06, 49M27

## 1 Introduction

We consider the composite minimization problem

$$
\begin{equation*}
\underset{\mathbf{x} \in \mathbf{H}}{\operatorname{minimize}} F(\mathbf{x}):=f(\mathbf{x})+g(\mathbf{x}), \quad g(\mathbf{x}):=\sum_{i=1}^{m} g_{i}\left(\mathbf{x}_{i}\right), \tag{1.1}
\end{equation*}
$$

where $\mathbf{H}$ is the direct sum of $m$ separable real Hilbert spaces $\left(\mathbf{H}_{i}\right)_{1 \leq i \leq m}$, that is, $\mathbf{H}=\bigoplus_{i=1}^{m} H_{i}$ and the following assumptions are satisfied unless stated otherwise.

A1 $f: \mathbf{H} \rightarrow \mathbb{R}$ is convex and differentiable.
A2 For every $\left.\left.i \in\{1, \cdots, m\}, g_{i}: \mathrm{H}_{i} \rightarrow\right]-\infty,+\infty\right]$ is proper convex and lower semicontinuous.
A3 For all $\mathbf{x} \in \mathbf{H}$ and $i \in\{1, \cdots, m\}$, the map $\nabla f\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{i-1}, \cdot, \mathrm{x}_{i+1}, \ldots, \mathrm{x}_{m}\right): \mathbf{H}_{i} \rightarrow \mathbf{H}$ is Lipschitz continuous with constant $L_{\text {res }}>0$ and the map $\nabla_{i} f\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{i-1}, \cdot, \mathrm{x}_{i+1}, \ldots, \mathrm{x}_{m}\right): \mathrm{H}_{i} \rightarrow \mathrm{H}_{i}$ is Lipschitz continuous with constant $L_{i}$. Note that $L_{\max }:=\max _{i} L_{i} \leq L_{\text {res }}$ and $L_{\min }:=\min _{i} L_{i}$.

A4 $F$ attains its minimum $F^{*}:=\min F$ on $\mathbf{H}$.

[^0]To solve problem 1.1, we use the following asynchronous block-coordinate descent algorithm. It is an extension of the parallel block-coordinate proximal gradient method considered in [41] to the asynchronous setting, where an inconsistent delayed gradient vector may be processed at each iteration.

Algorithm 1.1. Let $\left(i_{k}\right)_{k \in \mathbb{N}}$ be a sequence of i.i.d. random variables with values in $[m]:=\{1, \ldots, m\}$ and $\mathrm{p}_{i}$ be the probability of the event $\left\{i_{k}=i\right\}$, for every $i \in[m]$. Let $\left(\mathbf{d}^{k}\right)_{k \in \mathbb{N}}$ be a sequence of integer delay vectors, $\mathbf{d}^{k}=\left(\mathrm{d}_{1}^{k}, \ldots, \mathrm{~d}_{m}^{k}\right) \in \mathbb{N}^{m}$ such that $\max _{1 \leq i \leq m} \mathrm{~d}_{i}^{k} \leq \min \{k, \tau\}$ for some $\tau \in \mathbb{N}$. Let $\left(\gamma_{i}\right)_{1 \leq i \leq m} \in \mathbb{R}_{++}^{m}$ and $\boldsymbol{x}^{0}=\left(x_{1}^{0}, \ldots, x_{m}^{0}\right) \in \mathbf{H}$ be a constant random variable. Iterate

$$
\begin{align*}
& \text { for } k=0,1, \ldots \\
& \qquad \begin{array}{ll}
\text { for } i=1, \ldots, m \\
\left\lfloor x_{i}^{k+1}= \begin{cases}\operatorname{prox}_{\gamma_{i_{k}} g_{i_{k}}}\left(x_{i_{k}}^{k}-\gamma_{i_{k}} \nabla_{i_{k}} f\left(\boldsymbol{x}^{k-\mathbf{d}^{k}}\right)\right) & \text { if } i=i_{k} \\
x_{i}^{k} & \text { if } i \neq i_{k},\end{cases} \right.
\end{array}
\end{align*}
$$

where $\boldsymbol{x}^{k-\mathbf{d}^{k}}=\left(x_{1}^{k-\mathrm{d}_{1}^{k}}, \ldots, x_{m}^{k-\mathrm{d}_{m}^{k}}\right)$.
In this work, we assume the following stepsize rule

$$
\begin{equation*}
(\forall i \in[m]) \quad \gamma_{i}\left(L_{i}+2 \tau L_{\mathrm{res}} \mathbf{p}_{\max } / \sqrt{\mathbf{p}_{\min }}\right)<2, \tag{1.3}
\end{equation*}
$$

where $\mathrm{p}_{\text {max }}:=\max _{1 \leq i \leq m} \mathrm{p}_{i}$ and $\mathrm{p}_{\text {min }}:=\min _{1 \leq i \leq m} \mathrm{p}_{i}$. If there is no delay, namely $\tau=0$, the usual stepsize rule $\gamma_{i}<2 / L_{i}$ is obtained [14, 42].

The presence of the delay vectors in the above algorithm allows to describe a parallel computational model on multiple cores, as we explain below.

### 1.1 Asynchronous models

In this section we discuss an example of a parallel computational model, occurring in shared-memory system architectures, which can be covered by the proposed algorithm. Consider a situation where we have a machine with multiple cores. They all have access to a shared data $\boldsymbol{x}=\left(x_{1}, \ldots, x_{m}\right)$ and each core updates a block-coordinate $x_{i}, i \in[m]$, asynchronously without waiting for the others. The iteration's counter $k$ is increased any time a component of $\boldsymbol{x}$ is updated. When a core is given a coordinate to update, it has to read from the shared memory and compute a partial gradient. While performing these two operations, the data $x$ may have been updated by other cores. So, when the core is updating its assigned coordinate at iteration $k$, the gradient might no longer be up to date. This phenomenon is modelled by using a delay vector $\mathbf{d}^{k}$ and evaluating the partial gradient at $\boldsymbol{x}^{k-\mathbf{d}^{k}}$ as in Algorithm 1.1. Each component of the delay vector reflects how many times the corresponding coordinate of $\boldsymbol{x}$ have been updated since the core has read this particular coordinate from the shared memory. Note that different delays among the coordinates may arise since the shared data may be updated during the reading phase, so that the partial gradient ultimately is computed at a point which may not be consistent with any past instance of the shared data. This situation is called inconsistent read [6]. By contrast, in a consistent read model [29, 38], a lock is put during the reading phase and the delay originates only while computing the partial gradient. The delay is the same for all the block-coordinates, so that the value read by any core is a past instance of the shared data.

We remark that, in our setting, for all $k \in \mathbb{N}$, the delay vector $\mathbf{d}^{k}$ is considered to be a parameter that does not dependent on the random variable $i_{k}$, similarly to the works [30, 29, 16, 22]. Some
papers consider the case where the delay vector is a stochastic variable that may depend on $i_{k}[44,8]$ or that it is unbounded [44, 22]. A completely deterministic model, both in the block's selection and delays is studied in [12].

### 1.2 Related work

The topic on parallel asynchronous algorithm is not a recent one. In 1969, Chazan and Miranker [9] presented an asynchronous method for solving linear equations. Later on, Bertsekas and Tsitsiklis [6] proposed an inconsistent read model of asynchronous computation. Due to the availability of large amount of data and the importance of large scale optimization, in recent years we have witnessed a surge of interest in asynchronous algorithms. They have been studied and adapted to many optimization problems and methods such as stochastic gradient descent [1, 38, 19, 39, 28], randomized Kaczmarz algorithm [31], and stochastic coordinate descent [2, 29, 40, 50, 44].

In general, stochastic algorithms can be divided in two classes. The first one is when the function $f$ is an expectation i.e., $f(\mathrm{x})=\mathbb{E}[h(\mathrm{x} ; \xi)]$. At each iteration $k$ only a stochastic gradient $\nabla h\left(\cdot ; \xi_{k}\right)$ is computed based on the current sample $\xi_{k}$. In this setting, many asynchronous versions have been proposed, where delayed stochastic gradients are considered, see [35, 19, 3, 10, 27, 33]. The second class, which is the one we studied, is that of randomized block-coordinate methods. Below we describe the related literature.
[30] studied a problem and a model of asynchronicity which is similar to ours, but the proposed algorithm AsySPCD requires that the random variables $\left(i_{k}\right)_{k \in \mathbb{N}}$ are uniformly distributed (i.e, $p_{i}=$ $1 / m$ ) and that the stepsize is the same for all the block-coordinates. This latter assumption is an important limitation, since it does not exploit the possibility of adapting the stepsizes to the blockLipschitz constants of the partial gradients, hence allowing longer steps along block-coordinates. A linear rate of convergence is also obtained by exploiting a quadratic growth condition which is essentially equivalent to our error bound condition [18].

In the nonconvex case, [16] considers an asynchronous algorithm which may select the blocks both in an almost cyclic manner or randomly with a uniform probability. In the latter case, it is proved that the cluster points of the sequence of the iterates are almost surely stationary points of the objective function. However, the convergence of the whole sequence is not provided, nor is given any rate of convergence for the function values. Moreover, under the Kurdyka-Łojasiewicz (KL) condition $[18,7]$, linear convergence is also derived, but it is restricted to the deterministic case.

To conclude, we note that our results, when specialized to the case of zero delays, fully recover the ones given in [41].

### 1.3 Contributions

The main contributions of this work are summarized below:

- We first prove the almost sure weak convergence of the iterates $\left(x^{k}\right)_{k \in \mathbb{N}}$, generated by Algorithm 1.1, to a random variable $\boldsymbol{x}^{*}$ taking values in $\operatorname{argmin} F$. At the same time, we prove a sublinear rate of convergence of the function values in expectation, i.e, $\mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)\right]-\min F=o(1 / k)$. We also provide for the same quantity an explicit rate of $\mathcal{O}(1 / k)$, see Theorem 3.1.
- Under an error bound condition of Luo-Tseng type, on top of the strong convergence a.s of the iterates, we prove linear convergence in expectation of the function values and in mean of the iterates, see Theorem 4.2.

We improve the state-of-the-art under several aspects: we consider an arbitrary probability for the selection of the blocks; the adopted stepsize rule improves over the existing ones, and coincides with the one in [16] in the special case of uniform selection of the blocks - in particular, it allows for larger stepsizes when the number of blocks grows; the almost sure convergence of the iterates in the convex and stochastic setting is new and relies on a stochastic quasi-Fejerian analysis; linear convergence under an error bound condition is also new in the asynchronous stochastic scenario.

The rest of the paper is organized as follows. In the next subsection we set up basic notation. In Section 2 we recall few facts and we provide some preliminary results. The general convergence analysis is given in Section 3 where the main Theorem 3.1 is presented. Section 4 contains the convergence theory under an additional error bound condition, while applications are discussed in Section 5. The majority of proofs are postponed to Appendices A and B.

### 1.4 Notation

We set $\mathbb{R}_{+}=\left[0,+\infty\left[\right.\right.$ and $\left.\mathbb{R}_{++}=\right] 0,+\infty[$. For every integer $\ell \geq 1$ we define $[\ell]=\{1, \ldots, \ell\}$. For all $i \in[m]$, we denote indifferently the scalar products of $\mathbf{H}$ and $\mathbf{H}_{i}$ by $\langle\cdot, \cdot\rangle$ and:

$$
\left(\forall \mathbf{x}=\left(\mathrm{x}_{1}, \cdots, \mathrm{x}_{m}\right), \mathbf{y}=\left(\mathrm{y}_{1}, \cdots, \mathrm{y}_{m}\right) \in \mathbf{H}\right) \quad\langle\mathbf{x}, \mathbf{y}\rangle=\sum_{i=1}^{m}\left\langle\mathrm{x}_{i}, \mathrm{y}_{i}\right\rangle .
$$

$\|\cdot\|$ and $|\cdot|$ represent the norms associated to their scalar product in $\mathbf{H}$ and in any of $\mathrm{H}_{i}$ respectively. We also consider the canonical embedding, for all $i=1,2, \cdots, m, J_{i}: \mathrm{H}_{i} \rightarrow \mathbf{H}$, $\mathrm{x}_{i} \mapsto\left(0, \cdots, 0, \mathrm{x}_{i}, 0, \cdots, 0\right)$, with $x_{i}$ in the $i^{\text {th }}$ position. Random vectors and variables are defined on the underlying probability space $(\Omega, \mathfrak{A}, \mathrm{P})$. The default font is used for random variables while sans serif font is used for their realizations or deterministic variables. Let $\left(\alpha_{i}\right)_{1 \leq i \leq m} \in \mathbb{R}_{++}^{m}$. The direct sum operator $\mathrm{A}=\bigoplus_{i=1}^{m} \alpha_{i} \mathrm{Id}_{i}$, where $\mathrm{Id}_{i}$ is the identity operator on $\mathrm{H}_{i}$, is

$$
\begin{aligned}
\mathbf{A}: \mathbf{H} & \rightarrow \mathbf{H} \\
\mathbf{x}=\left(\mathrm{x}_{i}\right)_{1 \leq i \leq m} & \mapsto\left(\alpha_{i} \mathrm{x}_{i}\right)_{1 \leq i \leq m}
\end{aligned}
$$

This operator defines an equivalent scalar product on $\mathbf{H}$ as follows

$$
(\forall \mathbf{x} \in \mathbf{H})(\forall \mathbf{y} \in \mathbf{H}) \quad\langle\mathbf{x}, \mathbf{y}\rangle_{\mathrm{A}}=\langle\mathrm{A} \mathbf{x}, \mathbf{y}\rangle=\sum_{i=1}^{m} \alpha_{i}\left\langle\mathrm{x}_{i}, \mathrm{y}_{i}\right\rangle,
$$

which gives the norm $\|\mathbf{x}\|_{\mathrm{A}}^{2}=\sum_{i=1}^{m} \alpha_{i}\left|\mathrm{x}_{i}\right|^{2}$. We let

$$
\mathrm{V}=\bigoplus_{i=1}^{m} \mathrm{p}_{i} \mid \mathrm{ld}_{i}, \quad \mathrm{\Gamma}^{-1}=\bigoplus_{i=1}^{m} \frac{1}{\gamma_{i}} \mathrm{ld}_{i}, \quad \text { and } \quad \mathrm{W}=\bigoplus_{i=1}^{m} \frac{1}{\gamma_{i} \mathrm{p}_{i}} \mathrm{ld}_{i},
$$

where for all $i \in[m], \gamma_{i}$ and $\mathrm{p}_{i}$ are defined in Algorithm 1.1. We set $\mathrm{p}_{\text {max }}:=\max _{1 \leq i \leq m} \mathrm{p}_{i}$ and $\mathrm{p}_{\text {min }}:=\min _{1 \leq i \leq m} \mathrm{p}_{\mathrm{i}}$. If $\mathrm{S} \subset \mathbf{H}$ and $\mathbf{x} \in \mathbf{H}$, we set $\operatorname{dist}_{\mathrm{A}}(\mathbf{x}, \mathrm{S})=\inf _{\mathbf{z} \in \mathrm{S}}\|\mathbf{x}-\mathbf{z}\|_{\mathrm{A}}$. Let $\left.\left.\varphi: \mathbf{H} \rightarrow\right]-\infty,+\infty\right]$ be proper, convex, and lower semicontinuous. The domain of $\varphi$ is $\operatorname{dom} \varphi=\{\mathbf{x} \in \mathrm{H} \mid \varphi(\mathbf{x})<+\infty\}$ and the set of minimizers of $\varphi$ is $\operatorname{argmin} \varphi=\{\mathbf{x} \in \mathbf{H} \mid \varphi(\mathbf{x})=\inf \varphi\}$. If the function $\varphi: \mathbf{H} \rightarrow \mathbb{R}$ is differentiable, then for all $\mathbf{u}, \mathbf{x} \in \mathbf{H}$ and any symmetric positive definite operator $A$, we have $\left\langle\nabla^{\mathrm{A}} \varphi(\mathbf{x}), \mathbf{u}\right\rangle_{\mathrm{A}}=\langle\nabla \varphi(\mathbf{x}), \mathbf{u}\rangle$, where $\nabla^{\mathrm{A}}$ denotes the gradient operator in the norm $\|\cdot\|_{\mathrm{A}}$.

## 2 Preliminaries

In this section we present basic definitions and facts that are used in the rest of the paper. Most of them are already known, and we include them for clarity.

In the rest of the paper, we extend the definition of $x^{k}$ by setting $x^{k}=x^{0}$ for every $k \in$ $\{-\tau, \ldots,-1\}$. Using the notation of Algorithm 1.1, we also set, for any $k \in \mathbb{N}$

$$
\left\{\begin{align*}
\hat{\boldsymbol{x}}^{k} & =\boldsymbol{x}^{k-\mathbf{d}^{k}}  \tag{2.1}\\
\bar{x}_{i}^{k+1} & =\operatorname{prox}_{\gamma_{i} g_{i}}\left(x_{i}^{k}-\gamma_{i} \nabla_{i} f\left(\hat{\boldsymbol{x}}^{k}\right)\right) \text { for all } i \in[m] \\
\boldsymbol{x}^{k+1} & =\boldsymbol{x}^{k}+\mathrm{J}_{i_{k}}\left[\operatorname{prox}_{\gamma_{i_{k}} g_{i_{k}}}\left(x_{i_{k}}^{k}-\gamma_{i_{k}} \nabla_{i_{k}} f\left(\hat{\boldsymbol{x}}^{k}\right)\right)-x_{i_{k}}^{k}\right] \\
\boldsymbol{\Delta}^{k} & =\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1} .
\end{align*}\right.
$$

With this notation, we have

$$
\begin{equation*}
\bar{x}_{i_{k}}^{k+1}=\operatorname{prox}_{\gamma_{i_{k}} g_{i_{k}}}\left(x_{i_{k}}^{k}-\gamma_{i_{k}} \nabla_{i_{k}} f\left(\hat{\boldsymbol{x}}^{k}\right)\right)=x_{i_{k}}^{k+1} ; \quad \Delta_{i_{k}}^{k}=x_{i_{k}}^{k}-x_{i_{k}}^{k+1} . \tag{2.2}
\end{equation*}
$$

We remark that the random variables $\boldsymbol{x}^{k}$ and $\overline{\boldsymbol{x}}^{k+1}$ depend on the previously selected blocks, and related delays. More precisely, we have

$$
\begin{align*}
\boldsymbol{x}^{k} & =\boldsymbol{x}^{k}\left(i_{0}, \ldots, i_{k-1}, \mathbf{d}^{0}, \ldots, \mathbf{d}^{k-1}\right) \\
\overline{\boldsymbol{x}}^{k+1} & =\overline{\boldsymbol{x}}^{k+1}\left(i_{0}, \ldots, i_{k-1}, \mathbf{d}^{0}, \ldots, \mathbf{d}^{k}\right) . \tag{2.3}
\end{align*}
$$

From (2.1) and (2.2), we derive

$$
\begin{equation*}
\frac{x_{i_{k}}^{k}-x_{i_{k}}^{k+1}}{\gamma_{i_{k}}}-\nabla_{i_{k}} f\left(\hat{\boldsymbol{x}}^{k}\right) \in \partial g_{i_{k}}\left(x_{i_{k}}^{k+1}\right) \quad \text { and } \quad \frac{x_{i}^{k}-\bar{x}_{i}^{k+1}}{\gamma_{i}}-\nabla_{i} f\left(\hat{\boldsymbol{x}}^{k}\right) \in \partial g_{i}\left(\bar{x}_{i}^{k+1}\right) \tag{2.4}
\end{equation*}
$$

and therefore, for every $\mathbf{x} \in \mathbf{H}$

$$
\begin{equation*}
\left\langle\nabla_{i_{k}} f\left(\hat{\boldsymbol{x}}^{k}\right)-\frac{\Delta_{i_{k}}^{k}}{\gamma_{i_{k}}}, x_{i_{k}}^{k+1}-\mathrm{x}_{i_{k}}\right\rangle+g_{i_{k}}\left(x_{i_{k}}^{k+1}\right)-g_{i_{k}}\left(\mathrm{x}_{i_{k}}\right) \leq 0 . \tag{2.5}
\end{equation*}
$$

Suppose that $\mathbf{x}$ and $\mathbf{x}^{\prime}$ in $\mathbf{H}$ differ only for one component, say that of index $i$, then it follows from Assumption A3 and the Descent Lemma [36, Lemma 1.2.3], that

$$
\begin{align*}
f\left(\mathbf{x}^{\prime}\right) & =f\left(\mathrm{x}_{1}, \ldots, \mathrm{x}_{i-1}, \mathrm{x}_{i}^{\prime}, \mathrm{x}_{i+1}, \cdots, \mathrm{x}_{m}\right) \\
& \leq f(\mathbf{x})+\left\langle\nabla_{i} f(\mathbf{x}), \mathrm{x}_{i}^{\prime}-\mathrm{x}_{i}\right\rangle+\frac{L_{i}}{2}\left|\mathrm{x}_{i}^{\prime}-\mathrm{x}_{i}\right|^{2}  \tag{2.6}\\
& \leq f(\mathbf{x})+\left\langle\nabla f(\mathbf{x}), \mathbf{x}^{\prime}-\mathbf{x}\right\rangle+\frac{L_{\max }}{2}\left\|\mathbf{x}^{\prime}-\mathbf{x}\right\|^{2} . \tag{2.7}
\end{align*}
$$

We finally need the following results on the convergence of stochastic quasi-Fejér sequences and monotone summable positives sequences.

Fact 2.1 ([13], Proposition 2.3). Let S be a nonempty closed subset of a real Hilbert space H. Let $\mathscr{F}=\left(\mathcal{F}_{n}\right)_{n \in \mathbb{N}}$ be a sequence of sub-sigma algebras of $\mathcal{F}$ such that $(\forall n \in \mathbb{N}) \mathcal{F}_{n} \subset \mathcal{F}_{n+1}$. We denote by $\ell_{+}(\mathscr{F})$ the set of sequences of $\mathbb{R}_{+}$-valued random variables $\left(\xi_{n}\right)_{n \in \mathbb{N}}$ such that, for every $n \in \mathbb{N}$, $\xi_{n}$ is $\mathcal{F}_{n}$-measurable. We set

$$
\ell_{+}^{1}(\mathscr{F})=\left\{\left(\xi_{n}\right)_{n \in \mathbb{N}} \in \ell_{+}(\mathscr{F}) \mid \sum_{n \in \mathbb{N}} \xi_{n}<+\infty \quad \text { P-a.s. }\right\} .
$$

Let $\left(x_{n}\right)_{n \in \mathbb{N}}$ be a sequence of $\mathbf{H}$-valued random variables. Suppose that, for every $\mathrm{z} \in \mathrm{S}$, there exist $\left(\chi_{n}(\mathrm{z})\right)_{n \in \mathbb{N}} \in \ell_{+}^{1}(\mathscr{X}),\left(\vartheta_{n}(\mathrm{z})\right)_{n \in \mathbb{N}} \in \ell_{+}(\mathscr{X})$, and $\left(\eta_{n}(\mathrm{z})\right)_{n \in \mathbb{N}} \in \ell_{+}^{1}(\mathscr{X})$ such that the stochastic quasiFéjer property is satisfied P-a.s.:

$$
(\forall n \in \mathbb{N}) \quad \mathrm{E}\left[\left\|x_{n+1}-\mathrm{z}\right\|^{2} \mid \mathcal{F}_{n}\right]+\vartheta_{n}(\mathrm{z}) \leqslant\left(1+\chi_{n}(\mathrm{z})\right)\left\|x_{n}-\mathrm{z}\right\|^{2}+\eta_{n}(\mathrm{z}) .
$$

Then the following hold:
(i) $\left(x_{n}\right)_{n \in \mathbb{N}}$ is bounded P-a.s.
(ii) Suppose that the set of weak cluster points of the sequence $\left(x_{n}\right)_{n \in \mathbb{N}}$ is P -a.s. contained in S . Then $\left(x_{n}\right)_{n \in \mathbb{N}}$ weakly converges P -a.s. to an S -valued random variable.

Fact 2.2. Let $\left(a_{k}\right)_{k \in \mathbb{N}} \in \mathbb{R}_{+}^{\mathbb{N}}$ be a decreasing sequence of positive numbers and let $b \in \mathbb{R}_{+}$such that $\sum_{k \in \mathbb{N}} a_{k} \leq b<+\infty$. Then $a_{k}=o(1 /(k+1))$ and for every $k \in \mathbb{N}, a_{k} \leq b /(k+1)$.

Fact 2.3. $(\forall n, k \in \mathbb{Z}, k \geq n) \sum_{h=n}^{k-1} a_{h}=\sum_{h=n}^{k-1}(h-n+1) a_{h}-\sum_{h=n+1}^{k}(h-n) a_{h}+(k-n) a_{k}$.

### 2.1 Auxiliary lemmas

Here we collect technical lemmas needed for our analysis, using the notation given in (2.1). For reader's convenience, we provide all the proofs in Appendix A.

The following result appears in [30, page 357].
Lemma 2.4. Let $\left(\boldsymbol{x}_{k}\right)_{k \in \mathbb{N}}$ be the sequence generated by Algorithm 1.1. We have

$$
\begin{equation*}
(\forall k \in \mathbb{N}) \quad \boldsymbol{x}^{k}=\hat{\boldsymbol{x}}^{k}-\sum_{h \in J(k)}\left(\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right), \tag{2.8}
\end{equation*}
$$

where $J(k) \subset\{k-\tau, \ldots, k-1\}$ is a random set.
The next lemma bounds the difference between the delayed and the current gradient in terms of the steps along the block coordinates, see [30, equation A.7].

Lemma 2.5. Let $\left(x_{k}\right)_{k \in \mathbb{N}}$ be the sequence generated by Algorithm 1.1. It follows

$$
(\forall k \in \mathbb{N}) \quad\left\|\nabla f\left(\boldsymbol{x}^{k}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right)\right\| \leq L_{\mathrm{res}} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\| .
$$

Remark 2.6. Since $\|\cdot\|_{V}^{2} \leq \mathrm{p}_{\max }\|\cdot\|^{2}$ and $\|\cdot\|^{2} \leq \mathrm{p}_{\min }^{-1}\|\cdot\|_{V}^{2}$, Lemma 2.5 yields

$$
\begin{aligned}
\left\|\nabla f\left(\boldsymbol{x}^{k}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right)\right\| v & \leq \sqrt{\mathrm{p}_{\max }}\left\|\nabla f\left(\boldsymbol{x}^{k}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right)\right\| \\
& \leq L_{\mathrm{res}} \sqrt{\mathrm{p}_{\max }} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\| \\
& \leq L_{\mathrm{res}} \frac{\sqrt{\mathrm{p}_{\max }}}{\sqrt{\mathrm{p}_{\min }}} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\| \mathrm{v} .
\end{aligned}
$$

We set $L_{\text {res }}^{V}=L_{\text {res }} \frac{\sqrt{p_{\text {max }}}}{\sqrt{p_{\text {min }}}}$.

The result below yields a kind of inexact convexity inequality due to the presence of the delayed gradient vector.

Lemma 2.7. Let $\left(\boldsymbol{x}_{k}\right)_{k \in \mathbb{N}}$ be a sequence generated by Algorithm 1.1. Then, for every $k \in \mathbb{N}$,

$$
(\forall \mathbf{x} \in \mathbf{H}) \quad\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \mathbf{x}-\boldsymbol{x}^{k}\right\rangle \leq f(\mathbf{x})-f\left(\boldsymbol{x}^{k}\right)+\frac{\tau L_{\mathrm{res}}}{2} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2} .
$$

Lemma 2.8. Let $\mathbf{H}$ be a real Hilbert space. Let $\varphi: \mathbf{H} \rightarrow \mathbb{R}$ be differentiable and convex, and $\psi: \mathbf{H} \rightarrow]-\infty,+\infty]$ be proper, lower semicontinuous and convex. Let $\mathbf{x}, \hat{\mathbf{x}} \in \mathbf{H}$ and set $\mathbf{x}^{+}=$ $\operatorname{prox}_{\psi}(\mathbf{x}-\nabla \varphi(\hat{\mathbf{x}}))$. Then, for every $\mathbf{z} \in \mathbf{H}$,

$$
\begin{aligned}
\left(\mathbf{x}-\mathbf{x}^{+}, \mathbf{z}-\mathbf{x}\right\rangle \leq & \psi(\mathbf{z})-\psi(\mathbf{x})+\langle\nabla \varphi(\hat{\mathbf{x}}), \mathbf{z}-\mathbf{x}\rangle \\
& +\psi(\mathbf{x})-\psi\left(\mathbf{x}^{+}\right)+\left\langle\nabla \varphi(\hat{\mathbf{x}}), \mathbf{x}-\mathbf{x}^{+}\right\rangle-\left\|\mathbf{x}-\mathbf{x}^{+}\right\|^{2} .
\end{aligned}
$$

## 3 Convergence analysis

In this section we assume just convexity of the objective function and we provide worst case convergence rate as well as almost sure weak convergence of the iterates.

Throughout the section we set

$$
\begin{equation*}
\delta=\max _{i \in[m]}\left(L_{i} \gamma_{i}+2 \gamma_{i} \tau L_{\mathrm{res}}^{\vee} \sqrt{\mathrm{p}_{\max }}\right)=\max _{i \in[\mathrm{~m}]}\left(L_{i} \gamma_{i}+2 \gamma_{i} \tau L_{\mathrm{res}} \frac{\mathrm{p}_{\max }}{\sqrt{\mathrm{p}_{\min }}}\right), \tag{3.1}
\end{equation*}
$$

where the constants $L_{i}$ 's and $L_{\text {res }}$ are defined in Assumption A3 and the constant $L_{\mathrm{res}}^{V}$ is defined in Remark 2.6. The main convergence theorem is as follows.

Theorem 3.1. Let $\left(x^{k}\right)_{k \in \mathbb{N}}$ be the sequence generated by Algorithm 1.1 and suppose that $\delta<2$. Then the following hold.
(i) The sequence $\left(\boldsymbol{x}^{k}\right)_{k \in \mathbb{N}}$ weakly converges P -a.s. to a random variable that takes values in $\operatorname{argmin} F$.
(ii) $\mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)\right]-F^{*}=o(1 / k)$. Furthermore, for every integer $k \geq 1$,

$$
\begin{aligned}
& \mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)\right]-F^{*} \leq \frac{1}{k}\left(\frac{\operatorname{dist}_{\mathrm{W}}^{2}\left(\boldsymbol{x}^{0}, \operatorname{argmin} F\right)}{2}+C\left(F\left(\boldsymbol{x}^{0}\right)-F^{*}\right)\right) \text {, } \\
& \text { where } C \leq \frac{\max \left\{1,(2-\delta)^{-1}\right\}}{\mathrm{p}_{\min }}-1+\tau \frac{1}{\sqrt{\mathrm{p}_{\min }}(2-\delta)}\left(1+\frac{\mathrm{p}_{\max }}{\sqrt{\mathrm{p}_{\min }}}\right) .
\end{aligned}
$$

## Remark 3.2.

(i) Theorem 3.1 extends classical results about the forward-backward algorithm to the asynchronous and stochastic block-coordinate setting. See [42] and reference therein. Moreover, we note that the above results, when specialized to the synchronous case, that is, $\tau=0$, yield exactly [41, Theorem 4.9]. The $o(1 / k)$ was also proven in [26].
(ii) The almost sure weak convergence of the iterates for the asynchronous stochastic forwardbackward algorithm is new. In general only convergence in value is provided or, in the nonconvex case, cluster points of the sequence of the iterates are proven to be almost surely stationary points [16, 8].
(iii) If we suppose that the random variables $\left(i_{k}\right)_{k \in \mathbb{N}}$ are uniformly distributed over $[m]$, the stepsize rule reduces to $\gamma_{i}<2 /\left(L_{i}+2 \tau L_{\text {res }} / \sqrt{m}\right)$, which agrees with that given in [16] and gets better when the number of blocks $m$ increases. In this case, we see that the effect of the delay on the stepsize rule is mitigated by the number of blocks. In [8] the stepsize is not adapted to the blockwise Lipschitz constants $L_{i}$ 's, but it is chosen for each block as $\gamma<1 /\left(L_{\text {res }}+\tau^{2} L_{\text {res }} / 2\right)$, leading, in general, to smaller stepsizes. In addition, this rule has a worse dependence on the delay $\tau$ and lacks of any dependence on the number of blocks.
(iv) The framework of [8] is nonconvex and considers more general types of algorithms, in the flavour of majorization-minimization approaches [23]. On the other hand the assumptions are stronger (in particular, they assume $F$ to be coercive) and the rate of convergence is given with respect to $\left\|x^{k}-\operatorname{prox}_{g}\left(x^{k}-\nabla f\left(x^{k}\right)\right)\right\|^{2}$, a quantity which is hard to relate to $F\left(x^{k}\right)-F^{*}$. They also prove that the cluster points of the sequence of the iterates are almost surely stationary points.
(v) The work [30] was among the first to study an asynchronous version of the randomized coordinate gradient descent method. There, the coordinates were selected at random with uniform probability and the stepsize was chosen the same for every coordinate. However, the stepsize was chosen to depend exponentially on $\tau$, i.e as $\mathcal{O}\left(1 / \rho^{\tau}\right)$ with $\rho>1$, which is much worse than our $\mathcal{O}(1 / \tau)$. The same problem affects the constant in front of the bound of the rate of convergence which indeed is of the form $\mathcal{O}\left(\rho^{\tau}\right)$.

Before giving the proof of Theorem 3.1, we present few preliminary results. The first one is a proposition showing that the function values are decreasing in expectation. The proof of this proposition, as well as those of the next intermediate results, are given in Appendix B.

Proposition 3.3. Assume that $\delta<2$ and let $\left(x^{k}\right)_{k \in \mathbb{N}}$ be the sequence generated by Algorithm 1.1. Then, for every $k \in \mathbb{N}$,

$$
\begin{equation*}
(2-\delta) \frac{\mathrm{p}_{\min }}{2}\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\|_{\Gamma_{-1}}^{2} \leq F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right] \text { P-a.s. }, \tag{3.2}
\end{equation*}
$$

where $\alpha_{k}=\frac{L_{\mathrm{res}}^{\mathrm{V}}}{2 \sqrt{\mathrm{p}_{\max }}} \sum_{h=k-\tau}^{k-1}(h-(k-\tau)+1)\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|_{\mathrm{V}}^{2}$.
Lemma 3.4. Let $\left(x^{k}\right)_{k \in \mathbb{N}}$ be the sequence generated by Algorithm 1.1. Then for every $k \in \mathbb{N}$, we have

$$
\left\langle\nabla f\left(\boldsymbol{x}^{k}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\rangle_{\vee} \leq \tau L_{\mathrm{res}}^{\vee} \sqrt{\mathrm{p}_{\max }} \sum_{i=0}^{m} \mathrm{p}_{i}\left|\bar{x}_{i}^{k+1}-x_{i}^{k}\right|^{2}+\alpha_{k}-\mathrm{E}\left[\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right],
$$

where $\alpha_{k}$ is defined in Proposition 3.3.
The next result exhibits the relationship between the norms $\|\cdot\|_{W}$ and $\|\cdot\|_{\Gamma-1}$.
Lemma 3.5. Let $\left(\boldsymbol{x}_{k}\right)_{k \in \mathbb{N}}$ be a sequence generated by Algorithm 1.1. Let $k \in \mathbb{N}$ and let $\boldsymbol{x}$ be an $\mathbf{H}$-valued random variable which is measurable w.r.t. $i_{1}, \ldots, i_{k-1}$. Then,

$$
\begin{equation*}
\mathrm{E}\left[\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}\right\|_{\mathrm{W}}^{2} \mid i_{0}, \ldots, i_{k-1}\right]-\left\|\boldsymbol{x}^{k}-\boldsymbol{x}\right\|_{\mathrm{W}}^{2}=\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}\right\|_{\Gamma-1}^{2}-\left\|\boldsymbol{x}^{k}-\boldsymbol{x}\right\|_{\Gamma-1}^{2} \tag{3.3}
\end{equation*}
$$

and $\mathrm{E}\left[\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|_{\mathrm{W}}^{2} \mid i_{0}, \ldots, i_{k-1}\right]=\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\|_{\Gamma-1}^{2}$.

Proposition 3.6. Let $\left(\boldsymbol{x}_{k}\right)_{k \in \mathbb{N}}$ be a sequence generated by Algorithm 1.1 and suppose that $\delta<2$. Let $\left(\overline{\boldsymbol{x}}^{k}\right)_{k \in \mathbb{N}}$ and $\left(\alpha_{k}\right)_{k \in \mathbb{N}}$ be defined as in (2.1) and in Proposition 3.3 respectively. Then, for every $k \in \mathbb{N}$,

$$
\begin{aligned}
(\forall \mathbf{x} \in \mathbf{H}) \quad\left\langle\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}, \mathbf{x}-\boldsymbol{x}^{k}\right\rangle_{\Gamma-1} \leq & \frac{1}{\mathrm{p}_{\min }}\left(F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right]\right) \\
& +F(\mathbf{x})-F\left(\boldsymbol{x}^{k}\right)+\frac{\tau L_{\mathrm{res}}}{2} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2} \\
& +\frac{\delta-2}{2}\left\|\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\|_{\Gamma-1}^{2}
\end{aligned}
$$

Next we state a proposition that we will use throughout the rest of this paper. It corresponds to [41, Proposition 4.4].

Proposition 3.7. Let $\left(\boldsymbol{x}_{k}\right)_{k \in \mathbb{N}}$ be a sequence generated by Algorithm 1.1 and suppose that $\delta<2$. Let $\left(\alpha_{k}\right)_{k \in \mathbb{N}}$ be defined as in Proposition 3.3. Then, for every $k \in \mathbb{N}$,

$$
\begin{align*}
&(\forall \mathbf{x} \in \mathbf{H}) \quad \mathrm{E}\left[\left\|\boldsymbol{x}^{k+1}-\mathbf{x}\right\|_{\mathrm{W}}^{2} \mid i_{0}, \ldots, i_{k-1}\right] \\
& \leq\left\|\boldsymbol{x}^{k}-\mathbf{x}\right\|_{\mathrm{W}}^{2} \\
&+\frac{2}{\mathrm{p}_{\min }}\left(\frac{(\delta-1)_{+}}{2-\delta}+1\right)\left(F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right]\right) \\
&+\tau L_{\mathrm{res}} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2} \\
&+2\left(F(\mathbf{x})-F\left(\boldsymbol{x}^{k}\right)\right) \tag{3.4}
\end{align*}
$$

In the following, we show a general inequality from which we derive simultaneously the convergence of the iterates and the rate of convergence in expectation of the function values.

Proposition 3.8. Let $\left(\boldsymbol{x}_{k}\right)_{k \in \mathbb{N}}$ be a sequence generated by Algorithm 1.1 and suppose that $\delta<2$. Let $\left(\alpha_{k}\right)_{k \in \mathbb{N}}$ be defined as in Proposition 3.3. Then, for all $\mathbf{x} \in \mathbf{H}$,

$$
\mathrm{E}\left[\left\|\boldsymbol{x}^{k+1}-\mathbf{x}\right\|_{\mathrm{W}}^{2} \mid i_{0}, \ldots, i_{k-1}\right] \leq\left\|\boldsymbol{x}^{k}-\mathbf{x}\right\|_{\mathrm{W}}^{2}+2\left(F(\mathbf{x})-\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right]\right)+\xi_{k}
$$

where $\left(\xi^{k}\right)_{k \in \mathbb{N}}$ is a sequence of positive random variables such that

$$
\begin{equation*}
\sum_{k \in \mathbb{N}} \mathrm{E}\left[\xi_{k}\right] \leq 2 C\left(F\left(\boldsymbol{x}^{0}\right)-F^{*}\right), \tag{3.5}
\end{equation*}
$$

whith $C \leq \frac{\max \left\{1,(2-\delta)^{-1}\right\}}{\mathrm{p}_{\min }}-1+\frac{\tau}{\sqrt{\mathrm{p}_{\min }}(2-\delta)}\left(1+\frac{\mathrm{p}_{\max }}{\sqrt{\mathrm{p}_{\min }}}\right)$.
Proposition 3.9. Let $\left(\boldsymbol{x}_{k}\right)_{k \in \mathbb{N}}$ be a sequence generated by Algorithm 1.1 and suppose that $\delta<2$. Let $\left(\overline{\boldsymbol{x}}^{k}\right)_{k \in \mathbb{N}}$ be defined as in (2.1). Then there exists a sequence of $\mathbf{H}$-valued random variables $\left(\boldsymbol{v}^{k}\right)_{k \in \mathbb{N}}$ such that the following assertions hold:
(i) $\boldsymbol{v}^{k} \in \partial F\left(\overline{\boldsymbol{x}}^{k+1}\right) \mathrm{P}-a . s$.
(ii) $\boldsymbol{v}^{k} \rightarrow 0$ and $\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1} \rightarrow 0$ P-a.s.

We are now ready to prove the main theorem.

Proof of Theorem 3.1. (i): It follows from Proposition 3.8 that

$$
(\forall \mathbf{x} \in \operatorname{argmin} F) \quad \mathrm{E}\left[\left\|\boldsymbol{x}^{k+1}-\mathbf{x}\right\|_{\mathrm{W}}^{2} \mid i_{0}, \ldots, i_{k-1}\right] \leq\left\|\boldsymbol{x}^{k}-\mathbf{x}\right\|_{\mathrm{W}}^{2}+\xi_{k}
$$

where $\left(\xi_{k}\right)_{k \in \mathbb{N}}$ is a sequence of positive random variable which is P -a.s. summable. Thus, the sequence $\left(x^{k}\right)_{k \in \mathbb{N}}$ is stochastic quasi-Fejér with respect to $\operatorname{argmin} F$ in the norm $\|\cdot\|_{\mathrm{W}}$ (which is equivalent to $\|\cdot\|)$. Then according to Fact 2.1 it is bounded P -a.s. We now prove that $\operatorname{argmin} F$ contains the weak cluster points of $\left(\boldsymbol{x}^{k}\right)_{k \in \mathbb{N}} \mathrm{P}$-a.s. Indeed, let $\Omega_{1} \subset \Omega$ with $\mathrm{P}\left(\Omega \backslash \Omega_{1}\right)=0$ be such that items (i) and (ii) of Proposition 3.9 hold. Let $\omega \in \Omega_{1}$ and let $\mathbf{x}$ be a weak cluster point of $\left(\boldsymbol{x}^{k}(\omega)\right)_{k \in \mathbb{N}}$. There exists a subsequence $\left(\boldsymbol{x}^{k_{q}}(\omega)\right)_{q \in \mathbb{N}}$ which weakly converges to $\mathbf{x}$. By Proposition 3.9, we have $\overline{\boldsymbol{x}}^{k_{q}+1}(\omega) \rightharpoonup \boldsymbol{x}$, $\boldsymbol{v}^{k_{q}+1}(\omega) \rightarrow 0$, and $\boldsymbol{v}^{k_{q}+1}(\omega) \in \partial(f+g)\left(\overline{\boldsymbol{x}}^{k_{q}+1}(\omega)\right)$. Thus, [34, Proposition 1.6 (demiclosedness of the graph of the subgradient)] yields $0 \in \partial F(\mathbf{x})$ and hence $\mathbf{x} \in \operatorname{argmin} F$. Therefore, again by Fact 2.1 we conclude that the sequence $\left(\boldsymbol{x}^{k}\right)_{k \in \mathbb{N}}$ weakly converges to a random variable that takes value in $\operatorname{argmin} F \mathrm{P}$-a.s.
(ii): Choose $\mathbf{x} \in \operatorname{argmin} F$ in Proposition 3.8 and then take the expectation. Then we get

$$
\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1}\right]-F^{*} \leq \frac{1}{2}\left(\mathrm{E}\left[\left\|\boldsymbol{x}^{k}-\mathbf{x}\right\|_{\mathrm{W}}^{2}\right]-\mathrm{E}\left[\left\|\boldsymbol{x}^{k+1}-\mathbf{x}\right\|_{\mathrm{W}}^{2}\right]\right)+\frac{1}{2} \mathrm{E}\left[\xi_{k}\right] .
$$

Since $\sum_{k \in \mathbb{N}}\left(\mathrm{E}\left[\left\|\boldsymbol{x}^{k}-\mathbf{x}\right\|_{\mathrm{W}}^{2}\right]-\mathrm{E}\left[\left\|\boldsymbol{x}^{k+1}-\mathbf{x}\right\|_{\mathrm{W}}^{2}\right]\right) \leq\left\|\boldsymbol{x}^{0}-\mathbf{x}\right\|_{\mathrm{W}}^{2}$, and recalling the bound on $\sum_{k \in \mathbb{N}} \mathrm{E}\left[\xi_{k}\right]$ in (3.5), we have

$$
\sum_{k \in \mathbb{N}}\left(\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1}\right]-F^{*}\right) \leq \frac{\left\|\boldsymbol{x}^{0}-\mathbf{x}\right\|_{\mathrm{W}}^{2}}{2}+C\left(F\left(\boldsymbol{x}^{0}\right)-F^{*}\right) .
$$

Thus, since, in virtue of Proposition 3.2, $\left(\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1}\right]-F^{*}\right)_{k \in \mathbb{N}}$ is decreasing, the statement follows from Fact 2.2, considering that $\alpha_{k} \geq 0$.

## 4 Linear convergence under error bound condition

In the previous section we get a sublinear rate of convergence. Here we show that with an additional assumption we can get a better convergence rate. Also, we derive a strong convergence of the iterates, improving the weak convergence proved in Theorem 3.1.

We will assume that the following Luo-Tseng error bound condition [32] holds on a subset $\mathrm{X} \subset \mathbf{H}$ (containing the iterates $\boldsymbol{x}^{k}$ ).

$$
\begin{equation*}
(\forall \mathbf{x} \in \mathbf{X}) \quad \operatorname{dist}_{\Gamma^{-1}}(\mathbf{x}, \operatorname{argmin} F) \leq C_{\mathbf{X}, \Gamma^{-1}}\left\|\mathbf{x}-\operatorname{prox}_{g}^{\Gamma^{-1}}\left(\mathbf{x}-\nabla^{\Gamma^{-1}} f(\mathbf{x})\right)\right\|_{\Gamma^{-1}} . \tag{4.1}
\end{equation*}
$$

Remark 4.1. We recall that the condition above is equivalent to the Kurdyka-Lojasiewicz property and the quadratic growth condition [18, 7, 41]. Any of these conditions can be used to prove linear convergence rates for various algorithms.

The following theorem is the main result of this section. Here, linear convergence of the function values and strong convergence of the iterates are ensured.

Theorem 4.2. Let $\left(\boldsymbol{x}^{k}\right)_{k \in \mathbb{N}}$ be generated by Algorithm 1.1 and suppose $\delta<2$ and that the error bound condition (4.1) holds with $\mathrm{X} \supset\left\{\boldsymbol{x}^{k} \mid k \in \mathbb{N}\right\}$ P-a.s. for some $C_{\mathrm{X}, \Gamma-1}>0$. Then for all $k \in \mathbb{N}$,
(i) $\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)-F^{*}\right] \leq\left(1-\frac{\mathrm{p}_{\min }}{\kappa+\theta}\right)^{\left\lfloor\frac{k+1}{\tau+1}\right\rfloor} \mathrm{E}\left[F\left(\boldsymbol{x}^{0}\right)-F^{*}\right]$,
where

$$
\begin{aligned}
& \kappa=1+\frac{\left(2 C_{\mathrm{X}, \Gamma-1}+\delta-2\right)_{+}}{2-\delta}=\max \left\{1, \frac{2 C_{\mathrm{X}, \Gamma-1}}{2-\delta}\right\} \\
& \theta=\frac{\tau L_{\mathrm{res}} \gamma_{\max }}{2-\delta}\left(\frac{\mathrm{p}_{\max }^{2}}{\sqrt{\mathrm{p}_{\min }}}+1\right) \leq \frac{\sqrt{\mathrm{p}_{\min }}}{\mathrm{p}_{\max }(2-\delta)}\left(\frac{\mathrm{p}_{\max }^{2}}{\sqrt{\mathrm{p}_{\min }}}+1\right) .
\end{aligned}
$$

(ii) The sequence $\left(\boldsymbol{x}^{k}\right)_{k \in \mathbb{N}}$ converges strongly P -a.s. to a random variable $\boldsymbol{x}^{*}$ that takes values in $\operatorname{argmin} F$ and $\mathrm{E}\left[\left\|\boldsymbol{x}^{k}-\boldsymbol{x}^{*}\right\|_{\Gamma_{-1}}\right]=\mathcal{O}\left(\left(1-\mathrm{p}_{\min } /(\kappa+\theta)\right)^{\left\lfloor\frac{k}{\tau+1}\right\rfloor / 2}\right)$.

Proof. (i): From Proposition 3.6 we have

$$
\begin{aligned}
& \frac{1}{\mathrm{p}_{\text {min }}} \mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1}-F\left(\boldsymbol{x}^{k}\right)-\alpha_{k} \mid i_{0}, \ldots, i_{k-1}\right] \\
& \leq\left\|\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\|_{\Gamma-1}\left\|\boldsymbol{x}^{k}-\mathbf{x}\right\|_{\Gamma-1} \\
& \quad+F(\mathbf{x})-F\left(\boldsymbol{x}^{k}\right)+\frac{\tau L_{\mathrm{res}}}{2} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2} \\
& \quad+\frac{\delta-2}{2}\left\|\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\|_{\Gamma^{-1}}^{2},
\end{aligned}
$$

where $\alpha_{k}=\left(L_{\mathrm{res}} /\left(2 \sqrt{\mathrm{p}_{\text {min }}}\right)\right) \sum_{h=k-\tau}^{k-1}(h-(k-\tau)+1)\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|_{\mathrm{V}}^{2}$. Now, taking $\mathbf{x} \in \operatorname{argmin} F$ and using the error bound condition 4.1 and equation 3.2, we obtain

$$
\begin{align*}
& \left.\left.\frac{1}{\mathrm{p}_{\min }} \mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1}-F\left(\boldsymbol{x}^{k}\right)\right)-\alpha_{k} \right\rvert\, i_{0}, \ldots, i_{k-1}\right] \\
& \leq \\
& \leq\left(C_{\mathrm{X}, \Gamma-1}+\frac{\delta-2}{2}\right)\left\|\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\|_{\Gamma-1}^{2} \\
& \quad-\left(F\left(\boldsymbol{x}^{k}\right)-F^{*}\right)+\frac{\tau L_{\mathrm{res}}}{2} \sum_{h=k-\tau}^{k-1}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2} \\
& \leq  \tag{4.2}\\
& \left.\left.\leq \frac{\left(2 C_{\mathrm{X}, \Gamma-1}+\delta-2\right)_{+}}{(2-\delta) \mathrm{p}_{\min }} \mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)\right)+\alpha_{k}-F\left(\boldsymbol{x}^{k+1}\right)-\alpha_{k+1} \right\rvert\, i_{0}, \ldots, i_{k-1}\right] \\
& \\
& \quad-\left(F\left(\boldsymbol{x}^{k}\right)-F^{*}\right)+\frac{\tau L_{\mathrm{res}}}{2} \sum_{h=k-\tau}^{k-1}\left\|\boldsymbol{x}^{h}-\overline{\boldsymbol{x}}^{h+1}\right\|^{2},
\end{align*}
$$

Adding and removing $F^{*}$ in both expectation yield

$$
\begin{align*}
& \kappa \mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1}-F^{*} \mid i_{0}, \ldots, i_{k-1}\right] \leq \kappa \mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-F^{*} \mid i_{0}, \ldots, i_{k-1}\right] \\
& +\frac{\tau L_{\text {res }} \gamma_{\max } \mathrm{p}_{\text {min }}}{2} \sum_{h=k-\tau}^{k-1}\left\|\boldsymbol{x}^{h}-\overline{\boldsymbol{x}}^{h+1}\right\|_{\Gamma_{-1}}^{2} \\
& -\mathbf{p}_{\text {min }}\left(F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-F^{*}\right)+\mathbf{p}_{\text {min }} \alpha_{k}, \tag{4.3}
\end{align*}
$$

where $\kappa=1+\left(2 C_{\mathrm{X}, \Gamma^{-1}}+\delta-2\right)_{+} /(2-\delta)$. Now, since $\|\cdot\|_{\mathrm{V}}^{2} \leq \gamma_{\max } \mathrm{p}_{\max }^{2}\|\cdot\|_{\mathrm{W}}^{2}$ we have

$$
\begin{align*}
\mathrm{E}\left[\alpha_{k}\right] & \leq \frac{\tau L_{\mathrm{res}} \gamma_{\max } \mathrm{p}_{\max }^{2}}{2 \sqrt{\mathrm{p}_{\min }}} \sum_{h=k-\tau}^{k-1} \mathrm{E}\left[\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|_{\mathrm{W}}^{2}\right] \\
& =\frac{\tau L_{\mathrm{res}} \gamma_{\max } \mathrm{p}_{\max }^{2}}{2 \sqrt{\mathrm{p}_{\min }}} \sum_{h=k-\tau}^{k-1} \mathrm{E}\left[\left\|\overline{\boldsymbol{x}}^{h+1}-\boldsymbol{x}^{h}\right\|_{\Gamma-1}^{2}\right], \tag{4.4}
\end{align*}
$$

where in the last equality we used Lemma 3.5. From (3.2), we have, for $k$ such that $k-\tau \geq 0$,

$$
\begin{align*}
\sum_{h=k-\tau}^{k-1} \mathrm{E}\left[\left\|\overline{\boldsymbol{x}}^{h+1}-\boldsymbol{x}^{h}\right\|_{\Gamma-1}^{2}\right] & \leq \frac{2}{(2-\delta) \mathfrak{p}_{\min }} \sum_{h=k-\tau}^{k-1} \mathrm{E}\left[F\left(\boldsymbol{x}^{h}\right)+\alpha_{h}\right]-\mathrm{E}\left[F\left(\boldsymbol{x}^{h+1}\right)+\alpha_{h+1}\right] \\
& =\frac{2}{(2-\delta) \mathfrak{p}_{\min }}\left(\mathrm{E}\left[F\left(\boldsymbol{x}^{k-\tau}\right)+\alpha_{k-\tau}\right]-\mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}\right]\right) \\
& \leq \frac{2}{(2-\delta) \mathfrak{p}_{\min }}\left(\mathrm{E}\left[F\left(\boldsymbol{x}^{k-\tau}\right)+\alpha_{k-\tau}\right]-\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1}\right]\right) \\
& =\frac{2}{(2-\delta) \mathrm{p}_{\min }}\left(\mathrm{E}\left[F\left(\boldsymbol{x}^{k-\tau}\right)+\alpha_{k-\tau}-F^{*}\right]-\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1}-F^{*}\right]\right) . \tag{4.5}
\end{align*}
$$

Because the sequence $\left(\mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}\right]\right)_{k \in \mathbb{N}}$ is decreasing, the transition from the second line to the third one is allowed. Using (4.4) and (4.5) in (4.3) with total expectation, we obtain

$$
\begin{align*}
(\kappa+\theta) \mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1}-F^{*}\right] \leq & \left(\kappa-\mathbf{p}_{\min }\right) \mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-F^{*}\right] \\
& +\theta \mathrm{E}\left[F\left(\boldsymbol{x}^{k-\tau}\right)+\alpha_{k-\tau}-F^{*}\right] \\
\leq & \left(\kappa-\operatorname{p}_{\min }\right) \mathrm{E}\left[F\left(\boldsymbol{x}^{k-\tau}\right)+\alpha_{k-\tau}-F^{*}\right] \\
& +\theta \mathrm{E}\left[F\left(\boldsymbol{x}^{k-\tau}\right)+\alpha_{k-\tau}-F^{*}\right] \\
= & \left(\kappa+\theta-\operatorname{p}_{\min }\right) \mathrm{E}\left[F\left(\boldsymbol{x}^{k-\tau}\right)+\alpha_{k-\tau}-F^{*}\right], \tag{4.6}
\end{align*}
$$

where $\theta=(2-\delta)^{-1}\left(\frac{\tau L_{\mathrm{res}} \gamma_{\max } \mathrm{p}_{\max }^{2}}{\sqrt{\mathrm{p}_{\min }}}+\tau L_{\mathrm{res}} \gamma_{\max }\right)=\tau L_{\mathrm{res}} \gamma_{\max }(2-\delta)^{-1}\left(\frac{\mathrm{p}_{\max }^{2}}{\sqrt{\mathrm{p}_{\min }}}+1\right)$. That means

$$
\begin{align*}
\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1}-F^{*}\right] & \leq\left(1-\frac{\mathrm{p}_{\min }}{\kappa+\theta}\right) \mathrm{E}\left[F\left(\boldsymbol{x}^{k-\tau}\right)+\alpha_{k-\tau}-F^{*}\right] \\
& \leq\left(1-\frac{\mathrm{p}_{\min }}{\kappa+\theta}\right)^{\left\lfloor\frac{k+1}{\tau+1}\right\rfloor} \mathrm{E}\left[F\left(\boldsymbol{x}^{0}\right)+\alpha_{0}-F^{*}\right] . \tag{4.7}
\end{align*}
$$

Now for $k<\tau,\left\lfloor\frac{k+1}{\tau+1}\right\rfloor=0$. Because $\left(\mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}\right]\right)_{k \in \mathbb{N}}$ is decreasing, we know that

$$
\begin{aligned}
\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1}-F^{*}\right] & \leq \mathrm{E}\left[F\left(\boldsymbol{x}^{0}\right)+\alpha_{0}-F^{*}\right] \\
& =\left(1-\frac{\mathrm{p}_{\min }}{\kappa+\theta}\right)^{\left\lfloor\frac{k+1}{\tau+1}\right\rfloor} \mathrm{E}\left[F\left(\boldsymbol{x}^{0}\right)+\alpha_{0}-F^{*}\right] .
\end{aligned}
$$

So (4.7) remains true. Also from (B.10), we have

$$
\theta \leq \frac{\sqrt{\mathrm{p}_{\min }}}{\mathrm{p}_{\max }}(2-\delta)^{-1}\left(\frac{\mathrm{p}_{\max }^{2}}{\sqrt{\mathrm{p}_{\min }}}+1\right) .
$$

(ii): From Jensen inequality, (3.2) and (4.7), we have

$$
\begin{align*}
\mathrm{E}\left[\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|_{\Gamma-1}\right] & \leq \sqrt{\mathrm{E}\left[\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|_{\Gamma-1}^{2}\right]} \\
& \leq \sqrt{\mathrm{E}\left[\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\|_{\Gamma-1}^{2}\right]} \\
& \leq \sqrt{\frac{2}{\mathrm{p}_{\min }(2-\delta)} \mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-F^{*}\right]} \\
& \leq \sqrt{\frac{2}{\mathrm{p}_{\min }(2-\delta)}\left(1-\frac{\mathrm{p}_{\min }}{\kappa+\theta}\right)^{\left\lfloor\frac{k}{\tau+1}\right\rfloor} \mathrm{E}\left[F\left(\boldsymbol{x}^{0}\right)+\alpha_{0}-F^{*}\right] .} \tag{4.8}
\end{align*}
$$

Since $1-\boldsymbol{p}_{\text {min }} /(\kappa+\theta)<1$,

$$
\mathrm{E}\left[\sum_{k \in \mathbb{N}}\left\|x^{k+1}-\boldsymbol{x}^{k}\right\|_{\Gamma^{-1}}\right]=\sum_{k \in \mathbb{N}} \mathrm{E}\left[\left\|x^{k+1}-\boldsymbol{x}^{k}\right\|_{\Gamma^{-1}}\right]<\infty .
$$

Therefore $\sum_{k \in \mathbb{N}}\left\|x^{k+1}-x^{k}\right\|_{\Gamma-1}<\infty$ P-a.s. This means the sequence $\left(x^{k}\right)_{k \in \mathbb{N}}$ is a Cauchy sequence $P$-a.s. By Theorem 3.1 (i), this sequence has accumulation points that take values in $\operatorname{argmin} F$. So it converges strongly P -a.s. to a random variable that takes values in $\operatorname{argmin} F$.

Now let $\rho=1-\mathrm{p}_{\text {min }} /(\kappa+\theta)$. For all $n \in \mathbb{N}$,

$$
\left\|x^{k+n}-x^{k}\right\|_{\Gamma^{-1}} \leq \sum_{i=0}^{n-1}\left\|x^{k+i+1}-x^{k+i}\right\|_{\Gamma^{-1}} \leq \sum_{i=0}^{\infty}\left\|x^{k+i+1}-\boldsymbol{x}^{k+i}\right\|_{\Gamma^{-1}} .
$$

Letting $n \rightarrow \infty$ and using (4.8), we get

$$
\begin{aligned}
\mathrm{E}\left[\left\|\boldsymbol{x}^{k}-\boldsymbol{x}^{*}\right\|_{\Gamma^{-1}}\right] & \leq\left(\frac{2}{\mathrm{p}_{\min }(2-\delta)} \mathrm{E}\left[F\left(\boldsymbol{x}^{0}\right)+\alpha_{0}-F^{*}\right]\right)^{1 / 2} \sum_{i=0}^{\infty} \rho^{\left\lfloor\frac{k+i}{\tau+1}\right\rfloor / 2} \\
& \leq\left(\frac{2}{\mathrm{p}_{\min }(2-\delta)} \mathrm{E}\left[F\left(\boldsymbol{x}^{0}\right)+\alpha_{0}-F^{*}\right]\right)^{1 / 2} \rho^{\left\lfloor\frac{k}{\tau+1}\right\rfloor / 2} \sum_{i=0}^{\infty} \rho^{\left\lfloor\frac{i}{\tau+1}\right\rfloor / 2} \\
& =\rho^{\left\lfloor\frac{k}{\tau+1}\right\rfloor / 2}\left(\frac{2}{\mathrm{p}_{\min }(2-\delta)} \mathrm{E}\left[F\left(\boldsymbol{x}^{0}\right)+\alpha_{0}-F^{*}\right]\right)^{1 / 2} \frac{\tau+1}{1-\rho^{1 / 2}} .
\end{aligned}
$$

## Remark 4.3.

(i) A linear convergence rate is also given in [30, Theorem 4.1] by assuming a quadratic growth condition instead of the error bound condition (4.1). Their rate depend on the stepsize which in general can be very small, as explained earlier in point (v) of Remark 3.2.
(ii) The error bound condition (4.1) is sometimes satisfied globally, meaning on $X=\operatorname{dom} F$, so that the condition $\mathrm{X} \supset\left\{\boldsymbol{x}^{k} \mid k \in \mathbb{N}\right\}$ P-a.s. required in Theorem 4.2 is clearly fulfilled. This is the case when $F$ is strongly convex or when $f$ is quadratic and $g$ is the indicator function of a polytope (see Remark 4.17(iv) in [41]). More often, for general convex objectives, the error bound condition (4.1) is satisfied on sublevel sets of $F$ (see [41, Remark 4.18]). Therefore, it is important to find conditions ensuring that the sequence $\left(x^{k}\right)_{k \in \mathbb{N}}$ remains in a sublevel set. The next results address this issue.

We first give an analogue of Lemma 3.4.
Lemma 4.4. Let $\left(x^{k}\right)_{k \in \mathbb{N}}$ be the sequence generated by Algorithm 1.1. Then, for every $k \in \mathbb{N}$,

$$
\left\langle\nabla f\left(\boldsymbol{x}^{k}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), x^{k+1}-x^{k}\right\rangle \leq \tau L_{\mathrm{res}}\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|^{2}+\tilde{\alpha}_{k}-\tilde{\alpha}_{k+1},
$$

with $\tilde{\alpha}_{k}=\left(L_{\mathrm{res}} / 2\right) \sum_{h=k-\tau}^{k-1}(h-(k-\tau)+1)\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|^{2}$.
Proof. Let $k \in \mathbb{N}$. We have, from Cauchy-Schwarz inequality, the Young inequality and Remark 2.5, that

$$
\begin{aligned}
\left\langle\nabla f\left(\boldsymbol{x}^{k}\right)\right. & \left.-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\rangle \\
& \leq L_{\mathrm{res}} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\| \\
& \leq \frac{1}{2}\left[\frac{L_{\mathrm{res}}^{2}}{s}\left(\sum_{h \in J(k)}\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|\right)^{2}+s\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|^{2}\right] \\
& \leq \frac{1}{2}\left[\frac{\tau L_{\mathrm{res}}^{2}}{s}\left(\sum_{h=k-\tau}^{k-1}\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|^{2}\right)+s\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|^{2}\right] \\
& =\frac{s}{2}\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|^{2}+\frac{\tau L_{\mathrm{res}}^{2}}{2 s} \sum_{h=k-\tau}^{k-1}\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|^{2} .
\end{aligned}
$$

Using the same decomposition of the last term as in Lemma 3.4, we get

$$
\begin{aligned}
\left\langle\nabla f\left(\boldsymbol{x}^{k}\right)-\right. & \left.\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\rangle \\
\leq & \frac{s}{2}\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|^{2}+\frac{\tau L_{\mathrm{res}}^{2}}{2 s} \sum_{h=k-\tau}^{k-1}(h-(k-\tau)+1)\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|^{2} \\
& -\frac{\tau L_{\mathrm{res}}^{2}}{2 s} \sum_{h=k-\tau+1}^{k}(h-(k-\tau))\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|^{2} \\
& +\frac{\tau^{2} L_{\mathrm{res}}^{2}}{2 s}\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|^{2} .
\end{aligned}
$$

So taking

$$
\tilde{\alpha}_{k}=\frac{\tau L_{\mathrm{res}}^{2}}{2 s} \sum_{h=k-\tau}^{k-1}(h-(k-\tau)+1)\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|^{2},
$$

we get

$$
\left\langle\nabla f\left(\boldsymbol{x}^{k}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \bar{x}^{k+1}-x^{k}\right\rangle \leq\left(\frac{s}{2}+\frac{\tau^{2} L_{\mathrm{res}}^{2}}{2 s}\right)\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\|^{2}+\tilde{\alpha}_{k}-\tilde{\alpha}_{k+1} .
$$

By minimizing $s \mapsto\left(s / 2+\tau^{2} L_{\text {res }}^{2} /(2 s)\right)$, we find $s=\tau L_{\text {res }}$. We then obtain

$$
\left\langle\nabla f\left(\boldsymbol{x}^{k}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), x^{k+1}-x^{k}\right\rangle \leq \tau L_{\mathrm{res}}\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|^{2}+\tilde{\alpha}_{k}-\tilde{\alpha}_{k+1},
$$

and the statement follows.

Proposition 4.5. Let $\left(\boldsymbol{x}^{k}\right)_{k \in \mathbb{N}}$ be the sequence generated by Algorithm 1.1. Then, for every $k \in \mathbb{N}$,

$$
\begin{equation*}
\left(\frac{1}{\gamma_{i_{k}}}-\frac{L_{i_{k}}}{2}-\tau L_{\mathrm{res}}\right)\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|^{2} \leq F\left(\boldsymbol{x}^{k}\right)+\tilde{\alpha}_{k}-\left(F\left(\boldsymbol{x}^{k+1}\right)+\tilde{\alpha}_{k+1}\right) \quad \text { P-a.s., } \tag{4.9}
\end{equation*}
$$

where $\tilde{\alpha}_{k}=\left(L_{\mathrm{res}} / 2\right) \sum_{h=k-\tau}^{k-1}(h-(k-\tau)+1)\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|^{2}$.
Proof. Using Lemma 4.4 in equation (B.3), we have

$$
\begin{aligned}
F\left(\boldsymbol{x}^{k+1}\right) & \leq F\left(\boldsymbol{x}^{k}\right)+\left\langle\nabla_{i_{k}} f\left(\boldsymbol{x}^{k}\right)-\nabla_{i_{k}} f\left(\hat{\boldsymbol{x}}^{k}\right), \bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right\rangle-\left(\frac{1}{\gamma_{i_{k}}}-\frac{L_{i_{k}}}{2}\right)\left|\dot{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right|^{2} \\
& =F\left(\boldsymbol{x}^{k}\right)+\left\langle\nabla f\left(\boldsymbol{x}^{k}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\rangle-\left(\frac{1}{\gamma_{i_{k}}}-\frac{L_{i_{k}}}{2}\right)\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|^{2} \\
& \leq F\left(\boldsymbol{x}^{k}\right)+\tilde{\alpha}_{k}-\tilde{\alpha}_{k+1}-\left(\frac{1}{\gamma_{i_{k}}}-\frac{L_{i_{k}}}{2}-\tau L_{\mathrm{res}}\right)\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|^{2} .
\end{aligned}
$$

So the statement follows.
Corollary 4.6. Let $\left(\boldsymbol{x}^{k}\right)_{k \in \mathbb{N}}$ be generated by Algorithm 1.1 with the $\gamma_{i}$ 's satisfying the following stepsize rule

$$
\begin{equation*}
(\forall i \in[m]) \quad \gamma_{i}<\frac{2}{L_{i}+2 \tau L_{\mathrm{res}}} . \tag{4.10}
\end{equation*}
$$

Then

$$
\begin{equation*}
(\forall k \in \mathbb{N}) \quad F\left(\boldsymbol{x}^{k}\right) \leq F\left(\mathbf{x}^{0}\right) \quad \text { P-a.s. } \tag{4.11}
\end{equation*}
$$

So if the error bound condition (4.1) holds on the sublevel set $\mathrm{X}=\left\{F \leq F\left(\mathbf{x}^{0}\right)\right\}$, then the assumptions of Theorem 4.2 are met.

Proof. The left hand side in (4.9) is positive and hence $\left(F\left(\boldsymbol{x}_{k}\right)+\tilde{\alpha}_{k}\right)_{k \in \mathbb{N}}$ is decreasing P-a.s. Therefore, we have, for every $k \in \mathbb{N}$

$$
F\left(\boldsymbol{x}^{k}\right) \leq F\left(\boldsymbol{x}^{k}\right)+\tilde{\alpha}_{k} \leq F\left(\boldsymbol{x}^{0}\right)+\tilde{\alpha}_{0}=F\left(\mathbf{x}^{0}\right) .
$$

Remark 4.7. The rule (4.10) yields stepsizes possibly smaller than the ones given in Theorem 3.1, which requires $\gamma_{i}<2 /\left(L_{i}+2 \tau L_{\text {res }} \mathrm{p}_{\text {max }} / \sqrt{\boldsymbol{p}_{\text {min }}}\right)$. Indeed this happens when $\mathrm{p}_{\max } / \sqrt{\boldsymbol{p}_{\text {min }}}<1$. For instance if the distribution is uniform, we have $p_{\max } / \sqrt{p_{\text {min }}}=1 / \sqrt{m}<1$ whenever $m \geq 2$. On the bright side, there may exist distributions for which $\mathrm{p}_{\max } / \sqrt{\mathrm{p}_{\text {min }}}>1$.

## 5 Applications

Here we present two problems where Algorithm 1.1 can be useful.

### 5.1 The Lasso problem

We start with the Lasso problem [46], also known as basis pursuit [11]. It is a least-squares regression problem with an $\ell_{1}$ regularizer which favors sparse solutions. More precisely, given $\mathrm{A} \in \mathbb{R}^{n \times m}$ and $\mathrm{b} \in \mathbb{R}^{n}$, one aims at solving the following problem

$$
\begin{equation*}
\underset{\mathbf{x} \in \mathbb{R}^{m}}{\operatorname{minimize}} \frac{1}{2}\|\mathbf{A} \mathbf{x}-\mathbf{b}\|_{2}^{2}+\lambda\|\mathbf{x}\|_{1} \quad(\lambda>0) \tag{5.1}
\end{equation*}
$$

We clearly fall in the framework of problem (1.1) with $f(\mathbf{x})=(1 / 2)\|\mathbf{A} \mathbf{x}-\mathbf{b}\|_{2}^{2}$ and $g_{i}\left(\mathrm{x}_{i}\right)=\lambda\left|\mathbf{x}_{i}\right|$. The assumptions A1, A2, A3 and A4 are also satisfied. In particular, here $L_{i}=\left\|a_{i}\right\|^{2}$, where $a_{i}$ is the $i$-th column of A, $L_{\mathrm{res}}=\|\mathrm{A}\|^{2}$ and $F=f+g$ attains its minimum.

The Lasso technique is used in many fields, especially for high-dimensional problems - among others it is worth mentioning statistics, signal processing, and inverse problems; see [4, 47, 24, 5, $17,45]$ and references therein. Since there is no closed form solution for this problem, many iterative algorithms have been proposed to solve it: forward-backward, accelerated (proximal) gradient descent, (proximal) block coordinate descent, etc. [15, 4, 37, 21, 48, 20]. In the same vein, applying Algorithm 1.1 to the Lasso problem (5.1) yields the iterative scheme:

$$
\begin{align*}
& \text { for } n=0,1, \ldots \\
& \qquad \begin{array}{ll}
\text { for } i=1, \ldots, m \\
\left\lfloor x_{i}^{k+1}= \begin{cases}\operatorname{soft}_{\lambda_{i_{k}}}\left(x_{i_{k}}^{k}-\gamma_{i_{k}} a_{i_{k}}^{\top}\left(\mathrm{A} \boldsymbol{x}^{k-\mathbf{d}^{k}}-\mathrm{b}\right)\right) & \text { if } i=i_{k} \\
x_{i}^{k} & \text { if } i \neq i_{k},\end{cases} \right.
\end{array} \tag{5.2}
\end{align*}
$$

where, for every $\rho>0$, soft ${ }_{\rho}: \mathbb{R} \rightarrow \mathbb{R}$ is the soft thresholding operator (with threshold $\rho$ ) [42]. Thanks to Theorem 3.1 we know that the iterates $\left(x^{k}\right)_{k \in \mathbb{N}}$ generated are weakly convergent and the function values have a convergence rate of $o(1 / k)$. On top of that the cost function of the Lasso problem (5.1) satisfies the error bound condition (4.1) on its sublevel sets [49, Theorem 2]. So, following Corollary 4.6 and Theorem 4.2 , the iterates converge strongly (a.s.) and linearly in mean, whenever $\gamma_{i}<2 /\left(L_{i}+2 \tau L_{\mathrm{res}}\right)$, for all $i \in[m]$.

### 5.2 Linear convergence of dual proximal gradient method

We consider the problem

$$
\begin{equation*}
\underset{\mathrm{x} \in \mathrm{H}}{\operatorname{minimize}} \sum_{i=1}^{m} \phi_{i}\left(A_{i} \mathrm{x}\right)+h(\mathrm{x}), \tag{5.3}
\end{equation*}
$$

where, for all $i \in[m], A_{i}: \mathrm{H} \rightarrow \mathrm{G}_{i}$ is a linear operator between Hilbert spaces, $\left.\left.\phi_{i}: \mathrm{G}_{i} \rightarrow\right]-\infty,+\infty\right]$ is proper convex and lower semicontinuous, and $h: \mathrm{H} \rightarrow]-\infty,+\infty]$ is proper lower semicontinuous and $\sigma$-strongly convex $(\sigma>0)$. The first term of the objective function may represent the empirical data loss and the second term the regularizer. This problem arises in many applications in machine learning, signal processing and statistical estimation, and is commonly called regularized empirical risk minimization [43]. It includes, for instance, ridge regression and (soft margin) support vector machines [43], more generally Tikhonov regularization [25, Section 5.3].

In the following we apply Algorithm 1.1 to the dual of problem (5.3). Below we provide details. Set $\mathbf{G}=\bigoplus_{i=1}^{m} \mathrm{G}_{i}$ and $\mathbf{u}=\left(\mathbf{u}_{1}, \mathbf{u}_{2}, \ldots, \mathbf{u}_{m}\right)$. Then, the dual of problem (5.3) is

$$
\begin{equation*}
\underset{\mathbf{u} \in \mathrm{G}}{\operatorname{minimize}} F(\mathbf{u})=h^{*}\left(-\sum_{i=1}^{m} A_{i}^{*} \mathbf{u}_{i}\right)+\sum_{i=1}^{m} \phi_{i}^{*}\left(\mathbf{u}_{i}\right), \tag{5.4}
\end{equation*}
$$

where, $A_{i}^{*}$ is the adjoint operator of $A_{i} h^{*}$ and $\phi_{i}^{*}$ are the Fenchel conjugates of $h$ and $\phi_{i}$ respectively. The link between the dual variable $\mathbf{u}$ and the primal variable x is given by the rule $\mathbf{u} \mapsto \nabla h^{*}\left(-\sum_{i=1}^{m} A_{i}^{*} \mathbf{u}_{i}\right)$. Since $h^{*}$ is $(1 / \sigma)$-Lipschitz smooth, the dual problem above is in the form of problem (1.1). Thus, Algorithm (1.1) applied to the dual problem (5.4) gives

$$
\begin{align*}
& \text { for } k=0,1, \ldots \\
& \text { for } i=1, \ldots, m \\
& \left\lfloor\mathrm{u}_{i}^{k+1}= \begin{cases}\operatorname{prox}_{\gamma_{i_{k}} \phi_{i_{k}}^{*}}\left(\mathrm{u}_{i_{k}}^{k}+\gamma_{i_{k}} A_{i_{k}} \nabla h^{*}\left(-\sum_{j=1}^{m} A_{j}^{*} \mathrm{u}_{j}^{k-\mathrm{d}_{j}^{k}}\right)\right. & \text { if } i=i_{k} \\
\mathrm{u}_{i}^{k} & \text { if } i \neq i_{k},\end{cases} \right. \tag{5.5}
\end{align*}
$$

Suppose that $\nabla h^{*}=B$ is a linear operator and that the delay vecotr $\mathbf{d}^{k}=\left(\mathrm{d}_{1}^{k}, \cdots, \mathrm{~d}_{m}^{k}\right)$ is uniform, that is, $\mathrm{d}_{i}^{k}=\mathrm{d}_{j}^{k}=\mathrm{d}^{k} \in \mathbb{N}$. Then, using the primal variable, the KKT condition $\mathrm{x}^{k}=\nabla h^{*}\left(-\sum_{j=1}^{m} A_{j}^{*} \mathrm{u}_{j}^{k}\right)=-\sum_{j=1}^{m} B A_{j}^{*} \mathrm{u}_{j}^{k}$, and the fact that $\mathbf{u}^{k+1}$ and $\mathbf{u}^{k}$ differ only on the $i_{k}$ component, the algorithm becomes

$$
\begin{align*}
& \text { for } k=0,1, \ldots \\
& \qquad \begin{array}{ll}
\text { for } i=1, \ldots, m \\
\mathrm{u}_{i}^{k+1}= \begin{cases}\operatorname{prox}_{\gamma_{i_{k}} \phi_{i_{k}}^{*}}\left(\mathrm{u}_{i_{k}}^{k}+\gamma_{i_{k}} A_{i_{k}} \mathrm{x}^{k-\mathrm{d}^{k}}\right) & \text { if } i=i_{k} \\
\mathrm{u}_{i}^{k} & \text { if } i \neq i_{k} .\end{cases} \\
\mathrm{x}^{k+1}=\mathrm{x}^{k}-B A_{i_{k}}^{*}\left(\mathrm{u}_{i_{k}}^{k+1}-\mathrm{u}_{i_{k}}^{k}\right) .
\end{array} \tag{5.6}
\end{align*}
$$

The above algorithm requires a lock during the update of the primal variable x . On the contrary, the update of the dual variable $\mathbf{u}$ is completely asynchronous without any lock as in the setting we studied in this paper. To get a better understanding of this aspect, we will expose a concrete example: the ridge regression.

### 5.2.1 Example: Ridge regression

The ridge regression is the following regularized least squares problem.

$$
\begin{equation*}
\underset{\mathrm{w} \in \mathrm{H}}{\operatorname{minimize}} \frac{1}{\lambda m} \sum_{i=1}^{m}\left(\mathrm{y}_{i}-\left\langle\mathrm{w}, \mathrm{x}_{i}\right\rangle\right)^{2}+\frac{1}{2}\|\mathrm{w}\|^{2} . \tag{5.7}
\end{equation*}
$$

Its dual problem is

$$
\underset{\mathbf{u} \in \mathbb{R}^{m}}{\operatorname{minimize}} \frac{1}{2}\left\langle\left(\mathrm{~K}+\lambda m \mathrm{Id}_{m}\right) \mathbf{u}, \mathbf{u}\right\rangle-\langle\mathbf{y}, \mathbf{u}\rangle,
$$

where $\mathrm{K}=\mathrm{XX}^{*}$ and $\mathrm{X}: \mathrm{H} \rightarrow \mathbb{R}^{m}$, with $\mathrm{Xw}=\left(\left\langle\mathrm{w}, \mathrm{x}_{i}\right\rangle\right)_{1 \leq i \leq m}$. We remark that, in this situation, $A_{i}=\left\langle\cdot, \mathrm{x}_{i}\right\rangle, A_{i}^{*}=\mathrm{x}_{i}$ and $B=\operatorname{Id}$. Let $\mathbf{d}^{k}=\left(\mathrm{d}^{k}, \mathrm{~d}^{k}, \cdots, \mathrm{~d}^{k}\right)$. With $\mathrm{w}^{k}=\mathrm{X}^{*} \mathbf{u}^{k}$ and considering that the non smooth part $g$ is null, the algorithm is given by

$$
\begin{align*}
& \text { for } k=0,1, \ldots \\
& \qquad \begin{array}{ll}
\text { for } i=1, \ldots, m \\
\mathrm{u}_{i}^{k+1}= \begin{cases}\mathrm{u}_{i_{k}}^{k}-\gamma_{i_{k}}\left(\left\langle\mathrm{x}_{i_{k}}, \mathrm{w}^{k-\mathrm{d}^{k}}\right\rangle+\lambda m \mathrm{u}_{i_{k}}^{k-\mathrm{d}^{k}}-\mathrm{y}_{i_{k}}\right) & \text { if } i=i_{k} \\
\mathrm{u}_{i}^{k} & \text { if } i \neq i_{k} .\end{cases} \\
\mathrm{w}^{k+1}=\mathrm{w}^{k}-\gamma_{i_{k}} \mathrm{x}_{i_{k}}\left(\mathrm{u}_{i_{k}}^{k+1}-\mathrm{u}_{i_{k}}^{k}\right) .
\end{array}
\end{align*}
$$

Remark 5.1. Now we will compare the above dual asynchronous algorithm to the asynchronous stochastic gradient descent (ASGD) [38, 1]. We note that (5.8) yields

$$
\begin{aligned}
\mathrm{w}^{k+1} & =\mathrm{w}^{k}-\gamma_{i_{k}} \mathrm{x}_{i_{k}}\left(\mathrm{u}_{i_{k}}^{k+1}-\mathrm{u}_{i_{k}}^{k}\right) \\
& =\mathrm{w}^{k}-\gamma_{i_{k}}\left(\left\langle\mathrm{x}_{i_{k}}, \mathrm{w}^{k-\mathrm{d}^{k}}\right) \mathrm{x}_{i_{k}}+\lambda m \mathrm{u}_{i_{k}}^{k-\mathrm{d}^{k}} \mathrm{x}_{i_{k}}-\mathrm{y}_{i_{k}} \mathrm{x}_{i_{k}}\right)
\end{aligned}
$$

Instead, applying asynchronous SGD to the primal problem (5.7) multiply by $\lambda m$, we get

$$
\mathrm{w}^{k+1}=\mathrm{w}^{k}-\gamma_{k}^{\prime}\left(\left\langle\mathrm{x}_{i_{k}}, \mathrm{w}^{k-\mathrm{d}^{k}}\right) \mathrm{x}_{i_{k}}+\lambda m \mathrm{w}^{k-\mathrm{d}^{k}}-\mathrm{y}_{i_{k}} \mathrm{x}_{i_{k}}\right) .
$$

We see that the only difference is the second term inside the parentheses in both updates. Indeed the term $\mathrm{w}^{k-\mathrm{d}^{k}}=\mathrm{X}^{*} \mathbf{u}^{k-\mathrm{d}^{k}}=\sum_{i=1}^{m} \mathbf{u}_{i}^{k-\mathrm{d}^{k}} \mathrm{x}_{i}$ in ASGD is replaced by only one summand $\mathrm{u}_{i_{k}}^{k-\mathrm{d}^{k}} \mathrm{x}_{i_{k}}$ in our algorithm. However, a major difference between the two approaches lies in the way the stepsize is set. Indeed, in ASGD, the stepsize $\gamma_{k}^{\prime}$ is chosen with respect to the operator norm of $\mathrm{K}+\lambda m \mathrm{ld}$ i.e., the Lipschitz constant of the full gradient of the primal objective function, see [1, Theorem 1]. By contrast, in algorithm (5.8), for all $i \in[m]$, the stepsizes $\gamma_{i}^{k}$ are chosen with respect to the Lipschitz constant of the partial derivatives of the dual objective function i.e., $\mathrm{K}_{i, i}+\lambda m$. Not only the latter are easier to compute, they also allow for possibly longer steps along the coordinates.

## Acknowledgements

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 861137. S.V. acknowledges the financial support of the European Research Council (grant SLING 819789), the AFOSR projects FA9550-17-1-0390 and BAAAFRL- AFOSR-2016-0007 (European Office of Aerospace Research and Development), and the EU H2020-MSCA-RISE project NoMADS - DLV-777826.

## Appendices

## A Proofs of the auxiliary Lemmas in Section 2

In this section, for reader's convenience, we provide detailed proofs of the Lemmas presented in Section 2 , even though they are mostly not original. They are adapted from or can be found, e.g., in [30, 41].

Proof of Lemma 2.4. Let $k \in \mathbb{N}$. Since, for every $i \in[m], \mathrm{d}_{i}^{k} \leq \min \{k, \tau\}$, we have

$$
\begin{align*}
\boldsymbol{x}^{k-\mathbf{d}^{k}}-\boldsymbol{x}^{k} & =\sum_{i=1}^{m} \mathrm{~J}_{i}\left(x_{i}^{k-\mathrm{d}_{i}^{k}}-x_{i}^{k}\right) \\
& =\sum_{i=1}^{m} \mathrm{~J}_{i}\left(\sum_{h=k-\mathrm{d}_{i}^{k}}^{k-1}\left(x_{i}^{h}-x_{i}^{h+1}\right)\right) \\
& =\sum_{i=1}^{m} \mathrm{~J}_{i}\left(\sum_{h=k-\tau}^{k-1} \delta_{h, i}\left(x_{i}^{h}-x_{i}^{h+1}\right)\right) \\
& =\sum_{h=k-\tau}^{k-1} \sum_{i=1}^{m} \mathrm{~J}_{i}\left(\delta_{h, i}\left(x_{i}^{h}-x_{i}^{h+1}\right)\right) . \tag{A.1}
\end{align*}
$$

where $\delta_{h, i}=1$ if $h \geq k-\mathrm{d}_{i}^{k}$ and $\delta_{h, i}=0$ if $h<k-\mathrm{d}_{i}^{k}$. Note that for any $h \in\{k-\tau, \ldots, k-1\}$, in the sum

$$
\sum_{i=1}^{m} \mathrm{~J}_{i}\left(\delta_{h, i}\left(x_{i}^{h}-x_{i}^{h+1}\right)\right)
$$

at most one summand is different from zero, because the difference between $x^{h}$ and $x^{h+1}$ is only in the $i_{h}$-th component. So

$$
\sum_{i=1}^{m} \mathrm{~J}_{i}\left(\delta_{h, i}\left(x_{i}^{h}-x_{i}^{h+1}\right)\right)= \begin{cases}\mathrm{J}_{i_{h}}\left(x_{i_{h}}^{h}-x_{i_{h}}^{h+1}\right)=\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1} & \text { if } h \geq k-\mathrm{d}_{i_{h}}^{k} \\ 0 & \text { if } h<k-\mathrm{d}_{i_{h}}^{k}\end{cases}
$$

Therefore setting $J(k)=\left\{h \in\{k-\tau, \ldots, k-1\} \mid h \geq k-\mathrm{d}_{i_{h}}^{k}\right\}$, (A.1) yields (2.8). Note that, since $i_{h}$ is a random variable, $J(k)$ is a random set in the sense that $J(k)(\omega)=\{h \in\{k-\tau, \ldots, k-1\} \mid h \geq$ $\left.k-\mathrm{d}_{i_{h}(\omega)}^{k}\right\}$.

Proof of Lemma 2.5. Let $k \in \mathbb{N}$, let $p=\operatorname{card}(J(k))$, and let $\left(h_{j}\right)_{1 \leq j \leq p}$ be the elements of $J(k)$ ordered in (strictly) increasing order. Then, from Lemma 2.4 we have

$$
\begin{equation*}
\boldsymbol{x}^{k}-\hat{\boldsymbol{x}}^{k}=\sum_{j=1}^{p}\left(\boldsymbol{x}^{h_{j}+1}-\boldsymbol{x}^{h_{j}}\right) . \tag{A.2}
\end{equation*}
$$

Let's set, for each $t \in\{0, \ldots, p\}$

$$
\hat{\boldsymbol{x}}^{k, t}=\hat{\boldsymbol{x}}^{k}+\sum_{j=1}^{t}\left(\boldsymbol{x}^{h_{j}+1}-\boldsymbol{x}^{h_{j}}\right) .
$$

Then it follows

$$
\hat{\boldsymbol{x}}^{k, 0}=\hat{\boldsymbol{x}}^{k}, \quad \hat{\boldsymbol{x}}^{k, p}=\boldsymbol{x}^{k}, \quad \text { and } \quad \forall t \geq 1 \quad \hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t-1}=\boldsymbol{x}^{h_{t}+1}-\boldsymbol{x}^{h_{t}} .
$$

Therefore

$$
\boldsymbol{x}^{k}-\hat{\boldsymbol{x}}^{k}=\sum_{t=1}^{p}\left(\hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t-1}\right)
$$

and $\hat{\boldsymbol{x}}^{k, t}, \hat{\boldsymbol{x}}^{k, t-1}$ differ only in the value of a component. Thus

$$
\begin{aligned}
\left\|\nabla f\left(\boldsymbol{x}^{k}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right)\right\| & =\left\|\sum_{t=1}^{p} \nabla f\left(\hat{\boldsymbol{x}}^{k, t}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k, t-1}\right)\right\| \\
& \leq \sum_{t=1}^{p}\left\|\nabla f\left(\hat{\boldsymbol{x}}^{k, t}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k, t-1}\right)\right\| \\
& \leq L_{\mathrm{res}} \sum_{t=1}^{p}\left\|\hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t-1}\right\| \\
& =L_{\mathrm{res}} \sum_{t=1}^{p}\left\|\boldsymbol{x}^{h_{t+1}}-\boldsymbol{x}^{h t}\right\| \\
& =L_{\mathrm{res}} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\| .
\end{aligned}
$$

from which the result follows.
Proof of Lemma 2.7. Let $k \in \mathbb{N}$ and $\mathbf{x} \in \mathbf{H}$. Then

$$
\begin{aligned}
\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \mathbf{x}-\boldsymbol{x}^{k}\right\rangle= & \left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \mathbf{x}-\hat{\boldsymbol{x}}^{k}\right\rangle+\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \hat{\boldsymbol{x}}^{k}-\boldsymbol{x}^{k}\right\rangle \\
= & \left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \mathbf{x}-\hat{\boldsymbol{x}}^{k}\right\rangle+\sum_{t=0}^{p-1}\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t+1}\right\rangle \\
= & \left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \mathbf{x}-\hat{\boldsymbol{x}}^{k}\right\rangle \\
& \quad+\sum_{t=0}^{p-1}\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k, t}\right), \hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t+1}\right\rangle+\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k, t}\right), \hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t+1}\right\rangle .
\end{aligned}
$$

Thanks to the convexity of $f$ and (2.7), it follows

$$
\begin{aligned}
\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \mathbf{x}-\boldsymbol{x}^{k}\right\rangle \leq & f(\mathbf{x})-f\left(\hat{\boldsymbol{x}}^{k}\right)+\sum_{t=0}^{p-1} f\left(\hat{\boldsymbol{x}}^{k, t}\right)-f\left(\hat{\boldsymbol{x}}^{k, t+1}\right)+\frac{L_{\max }}{2}\left\|\hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t+1}\right\|^{2} \\
& +\sum_{t=0}^{p-1}\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k, t}\right), \hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t+1}\right\rangle \\
= & f(\mathbf{x})-f\left(\boldsymbol{x}^{k}\right)+\frac{L_{\max }}{2} \sum_{t=0}^{p-1}\left\|\hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t+1}\right\|^{2}
\end{aligned}
$$

$$
\begin{aligned}
& \quad+\sum_{t=0}^{p-1} \sum_{s=0}^{t-1}\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k, s}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k, s+1}\right), \hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t+1}\right\rangle \\
& \leq f(\mathbf{x})-f\left(\boldsymbol{x}^{k}\right)+\frac{L_{\mathrm{max}}}{2} \sum_{t=0}^{p-1}\left\|\hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t+1}\right\|^{2} \\
& \quad+L_{\mathrm{res}} \sum_{t=0}^{p-1} \sum_{s=0}^{t-1}\left\|\hat{\boldsymbol{x}}^{k, s}-\hat{\boldsymbol{x}}^{k, s+1}\right\|\left\|\hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t+1}\right\| .
\end{aligned}
$$

Using the equality of the square of sum, Holder inequality and $L_{\text {max }} \leq L_{\text {res }}$, we finally get

$$
\begin{aligned}
\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \mathbf{x}-\boldsymbol{x}^{k}\right\rangle \leq & f(\mathbf{x})-f\left(\boldsymbol{x}^{k}\right)+\frac{L_{\mathrm{max}}}{2} \sum_{t=0}^{p-1}\left\|\hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t+1}\right\|^{2} \\
& +\frac{L_{\mathrm{res}}}{2}\left[\left(\sum_{t=0}^{p-1}\left\|\hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t+1}\right\|\right)^{2}-\sum_{t=0}^{p-1}\left\|\hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t+1}\right\|^{2}\right] \\
= & f(\mathbf{x})-f\left(\boldsymbol{x}^{k}\right)+\frac{L_{\mathrm{res}}}{2}\left(\sum_{t=0}^{p-1}\left\|\hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t+1}\right\|\right)^{2} \\
& +\left(\frac{L_{\mathrm{max}}}{2}-\frac{L_{\mathrm{res}}}{2}\right) \sum_{t=0}^{p-1}\left\|\hat{\boldsymbol{x}}^{k, t}-\hat{\boldsymbol{x}}^{k, t+1}\right\|^{2} \\
\leq & f(\mathbf{x})-f\left(\boldsymbol{x}^{k}\right)+\frac{\tau L_{\mathrm{res}}}{2} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2}
\end{aligned}
$$

The statement follows.
Proof of Lemma 2.8. Let $\mathbf{z} \in \mathbf{H}$. It follows from the definition of $\mathbf{x}^{+}$that $\mathbf{x}-\mathbf{x}^{+}-\nabla \varphi(\hat{\mathbf{x}}) \in \partial \psi\left(\mathbf{x}^{+}\right)$. Therefore, $\psi(\mathbf{z}) \geq \psi\left(\mathbf{x}^{+}\right)+\left\langle\mathbf{x}-\mathbf{x}^{+}-\nabla \varphi(\hat{\mathbf{x}}), \mathbf{z}-\mathbf{x}^{+}\right\rangle$, hence

$$
\left\langle\mathbf{x}-\mathbf{x}^{+}, \mathbf{z}-\mathbf{x}^{+}\right\rangle \leq \psi(\mathbf{z})-\psi\left(\mathbf{x}^{+}\right)+\left\langle\nabla \varphi(\hat{\mathbf{x}}), \mathbf{z}-\mathbf{x}^{+}\right\rangle .
$$

Then,

$$
\left\langle\mathbf{x}-\mathbf{x}^{+}, \mathbf{z}-\mathbf{x}\right\rangle+\left\langle\mathbf{x}-\mathbf{x}^{+}, \mathbf{x}-\mathbf{x}^{+}\right\rangle \leq \psi(\mathbf{z})-\psi\left(\mathbf{x}^{+}\right)+\langle\nabla \varphi(\hat{\mathbf{x}}), \mathbf{z}-\mathbf{x}\rangle+\left\langle\nabla \varphi(\hat{\mathbf{x}}), \mathbf{x}-\mathbf{x}^{+}\right\rangle
$$

Rearranging the terms the statement follows.

## B Proofs of Section 3

Proof of Lemma 3.4. Let $k \in \mathbb{N}$. We have, from Cauchy-Schwarz inequality, the Young inequality and Remark 2.5, that

$$
\begin{aligned}
\left\langle\nabla f\left(\boldsymbol{x}^{k}\right)\right. & \left.-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\rangle_{\mathrm{V}} \\
& \leq L_{\mathrm{res}}^{\vee} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\| \mathrm{\vee}\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\| \vee
\end{aligned}
$$

$$
\begin{aligned}
& \leq \frac{1}{2}\left[\frac{\left(L_{\mathrm{res}}^{\vee}\right)^{2}}{s}\left(\sum_{h \in J(k)}\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\| \mathrm{V}\right)^{2}+s\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\|_{\mathrm{V}}^{2}\right] \\
& \leq \frac{1}{2}\left[\frac{\tau\left(L_{\mathrm{res}}^{\vee}\right)^{2}}{s}\left(\sum_{h=k-\tau}^{k-1}\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|_{\mathrm{V}}^{2}\right)+s\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\|_{\mathrm{V}}^{2}\right] \\
& =\frac{s}{2}\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\|_{\mathrm{V}}^{2}+\frac{\tau\left(L_{\mathrm{res}}^{\vee}\right)^{2}}{2 s} \sum_{h=k-\tau}^{k-1}\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|_{\mathrm{V}}^{2}
\end{aligned}
$$

Now, thanks to a decomposition of the last term by Fact 2.3, we obtain

$$
\begin{aligned}
\left\langle\nabla f\left(\boldsymbol{x}^{k}\right)-\right. & \left.\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\rangle_{\mathrm{V}} \\
\leq & \frac{s}{2}\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\|_{\mathrm{V}}^{2}+\frac{\tau\left(L_{\mathrm{res}}^{\mathrm{V}}\right)^{2}}{2 s} \sum_{h=k-\tau}^{k-1}(h-(k-\tau)+1)\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|_{\mathrm{V}}^{2} \\
& -\frac{\tau\left(L_{\mathrm{res}}^{\vee}\right)^{2}}{2 s} \sum_{h=k-\tau+1}^{k}(h-(k-\tau))\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|_{\mathrm{V}}^{2} \\
& +\frac{\tau^{2}\left(L_{\mathrm{res}}^{\mathrm{V}}\right)^{2}}{2 s}\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|_{\mathrm{V}}^{2}
\end{aligned}
$$

We recall that $\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|_{\mathrm{V}}^{2}=\mathrm{p}_{i_{k}}\left|\bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right|^{2}$. So taking

$$
\alpha_{k}=\frac{\tau\left(L_{\mathrm{res}}^{\mathrm{V}}\right)^{2}}{2 s} \sum_{h=k-\tau}^{k-1}(h-(k-\tau)+1)\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|_{\mathrm{V}}^{2}
$$

we get

$$
\begin{aligned}
\mathrm{E}\left[\left\langle\nabla f\left(\boldsymbol{x}^{k}\right)\right.\right. & \left.\left.-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \bar{x}^{k+1}-x^{k}\right\rangle \vee \mid i_{0}, \ldots, i_{k-1}\right] \\
& \leq \frac{s}{2}\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\|_{\mathrm{V}}^{2}+\frac{\tau^{2}\left(L_{\mathrm{res}}^{\vee}\right)^{2}}{2 s} \sum_{i=0}^{m} \mathrm{p}_{i}^{2}\left|\bar{x}_{i}^{k+1}-x_{i}^{k}\right|^{2}+\alpha_{k}-\mathrm{E}\left[\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right]
\end{aligned}
$$

Meaning

$$
\begin{aligned}
\left\langle\nabla f\left(\boldsymbol{x}^{k}\right)\right. & \left.-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \bar{x}^{k+1}-x^{k}\right\rangle_{\mathrm{V}} \\
& \leq \sum_{i=0}^{m} \mathrm{p}_{i}\left(\frac{s}{2}+\frac{\tau^{2}\left(L_{\mathrm{res}}^{\mathrm{V}}\right)^{2}}{2 s} \mathrm{p}_{i}\right)\left|\bar{x}_{i}^{k+1}-x_{i}^{k}\right|^{2}+\alpha_{k}-\mathrm{E}\left[\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right] \\
& \leq \sum_{i=0}^{m} \mathrm{p}_{i}\left(\frac{s}{2}+\frac{\tau^{2}\left(L_{\mathrm{res}}^{\mathrm{V}}\right)^{2}}{2 s} \mathrm{p}_{\max }\right)\left|\bar{x}_{i}^{k+1}-x_{i}^{k}\right|^{2}+\alpha_{k}-\mathrm{E}\left[\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right]
\end{aligned}
$$

By minimizing $s \mapsto\left(\frac{s}{2}+\frac{\tau^{2}\left(L_{\mathrm{res}}^{\mathrm{V}}\right)^{2}}{2 s} \mathrm{p}_{\max }\right)$, we find $s=\tau L_{\mathrm{res}}^{\mathrm{V}} \sqrt{\mathrm{p}_{\max }}$. We then get

$$
\begin{aligned}
\left\langle\nabla f\left(\boldsymbol{x}^{k}\right)\right. & \left.-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \bar{x}^{k+1}-x^{k}\right\rangle_{\mathrm{V}} \\
& \leq \tau L_{\mathrm{res}}^{\vee} \sqrt{\mathrm{p}_{\max }} \sum_{i=0}^{m} \mathrm{p}_{i}\left|\bar{x}_{i}^{k+1}-x_{i}^{k}\right|^{2}+\alpha_{k}-\mathrm{E}\left[\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right]
\end{aligned}
$$

and $\alpha_{k}=\frac{L_{\mathrm{res}}^{\mathrm{V}}}{2 \sqrt{\mathrm{p}_{\max }}} \sum_{h=k-\tau}^{k-1}(h-(k-\tau)+1)\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|_{\mathrm{V}}^{2}$.
Proof of Lemma 3.5. We have

$$
\begin{equation*}
\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}\right\|_{\mathrm{W}}^{2}=\sum_{i=1}^{m} \frac{1}{\mathrm{p}_{i} \gamma_{i}}\left|x_{i}^{k+1}-x_{i}\right|^{2}=\frac{1}{\mathrm{p}_{i_{k}} \gamma_{i_{k}}}\left|\bar{x}_{i_{k}}^{k+1}-x_{i_{k}}\right|^{2}+\left\|\boldsymbol{x}^{k}-\boldsymbol{x}\right\|_{\mathrm{W}}^{2}-\frac{1}{\mathrm{p}_{i_{k}} \gamma_{i_{k}}}\left|x_{i_{k}}^{k}-x_{i_{k}}\right|^{2} . \tag{B.1}
\end{equation*}
$$

Thus, taking the conditional expectation we have

$$
\begin{equation*}
\mathrm{E}\left[\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}\right\|_{\mathrm{W}}^{2} \mid i_{0}, \ldots, i_{k-1}\right]=\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}\right\|_{\Gamma_{-1}}^{2}+\left\|\boldsymbol{x}^{k}-\boldsymbol{x}\right\|_{\mathrm{W}}^{2}-\left\|\boldsymbol{x}^{k}-\boldsymbol{x}\right\|_{\Gamma-1}^{2} \tag{B.2}
\end{equation*}
$$

and (3.3) follows. The second equation follows from (3.3), by choosing $\boldsymbol{x}=\boldsymbol{x}^{k}$.
Proof of Proposition 3.3. Let $k \in \mathbb{N}$. We have from the descent lemma along the $i_{k}$-th blockcoordinate,

$$
\begin{aligned}
F\left(\boldsymbol{x}^{k+1}\right) \leq & f\left(\boldsymbol{x}^{k}\right)+\left\langle\nabla_{i_{k}} f\left(\boldsymbol{x}^{k}\right), \bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right\rangle+\frac{L_{i_{k}}}{2}\left|\bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right|^{2}+\sum_{i=1}^{n} g_{i}\left(x_{i}^{k+1}\right) \\
= & f\left(\boldsymbol{x}^{k}\right)+\left\langle\nabla_{i_{k}} f\left(\boldsymbol{x}^{k}\right), \bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right\rangle+\frac{L_{i_{k}}}{2}\left|\bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right|^{2}+\left(g_{i_{k}}\left(x_{i_{k}}^{k+1}\right)+\sum_{i \neq i_{k}}^{n} g_{i}\left(x_{i}^{k}\right)\right) \\
= & f\left(\boldsymbol{x}^{k}\right)+\left\langle\nabla_{i_{k}} f\left(\boldsymbol{x}^{k}\right), \bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right\rangle+\frac{L_{i_{k}}}{2}\left|\bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right|^{2} \\
& \quad+\left(g_{i_{k}}\left(x_{i_{k}}^{k+1}\right)-g_{i_{k}}\left(x_{i_{k}}^{k}\right)+g\left(\boldsymbol{x}^{k}\right)\right) \\
= & F\left(\boldsymbol{x}^{k}\right)+\left\langle\nabla_{i_{k}} f\left(\boldsymbol{x}^{k}\right), \bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right\rangle+\frac{L_{i_{k}}}{2}\left|\bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right|^{2}+\left(g_{i_{k}}\left(\bar{x}_{i_{k}}^{k+1}\right)-g_{i_{k}}\left(x_{i_{k}}^{k}\right)\right) \\
= & F\left(\boldsymbol{x}^{k}\right)+\left\langle\nabla_{i_{k}} f\left(\boldsymbol{x}^{k}\right)-\nabla_{i_{k}} f\left(\hat{\boldsymbol{x}}^{k}\right), \bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right\rangle+\frac{L_{i_{k}}}{2}\left|\bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right|^{2} \\
& \quad+\left(\left\langle\nabla_{i_{k}} f\left(\hat{\boldsymbol{x}}^{k}\right), \bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right\rangle+g_{i_{k}}\left(\bar{x}_{i_{k}}^{k+1}\right)-g_{i_{k}}\left(x_{i_{k}}^{k}\right)\right) .
\end{aligned}
$$

From (2.5), we can write that

$$
\begin{equation*}
F\left(\boldsymbol{x}^{k+1}\right) \leq F\left(\boldsymbol{x}^{k}\right)+\left\langle\nabla_{i_{k}} f\left(\boldsymbol{x}^{k}\right)-\nabla_{i_{k}} f\left(\hat{\boldsymbol{x}}^{k}\right), \bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right\rangle-\left(\frac{1}{\gamma_{i_{k}}}-\frac{L_{i_{k}}}{2}\right)\left|\bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right|^{2} \tag{B.3}
\end{equation*}
$$

By taking the conditional expectation, it follows:

$$
\begin{aligned}
& \mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right) \mid i_{0}, \ldots, i_{k-1}\right] \\
& \leq F\left(\boldsymbol{x}^{k}\right)+\mathrm{E}\left[\left\langle\nabla_{i_{k}} f\left(\boldsymbol{x}^{k}\right)-\nabla_{i_{k}} f\left(\hat{\boldsymbol{x}}^{k}\right), \bar{x}_{i_{k}}^{k+1}-x_{i_{k}}^{k}\right\rangle \mid i_{0}, \ldots, i_{k-1}\right] \\
& -\sum_{i=0}^{m} \mathrm{p}_{i}\left(\frac{1}{\gamma_{i}}-\frac{L_{i}}{2}\right)\left|\bar{x}_{i}^{k+1}-x_{i}^{k}\right|^{2} \\
& =F\left(\boldsymbol{x}^{k}\right)+\sum_{i=0}^{m} \mathrm{p}_{i}\left\langle\nabla_{i} f\left(\boldsymbol{x}^{k}\right)-\nabla_{i} f\left(\hat{\boldsymbol{x}}^{k}\right), \bar{x}_{i}^{k+1}-x_{i}^{k}\right\rangle \\
& -\sum_{i=0}^{m} \mathrm{p}_{i}\left(\frac{1}{\gamma_{i}}-\frac{L_{i}}{2}\right)\left|\bar{x}_{i}^{k+1}-x_{i}^{k}\right|^{2}
\end{aligned}
$$

$$
\begin{align*}
& =F\left(\boldsymbol{x}^{k}\right)+\left\langle\nabla f\left(\boldsymbol{x}^{k}\right)-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\rangle_{\vee} \\
& \quad-\sum_{i=0}^{m} \mathrm{p}_{i}\left(\frac{1}{\gamma_{i}}-\frac{L_{i}}{2}\right)\left|\bar{x}_{i}^{k+1}-x_{i}^{k}\right|^{2} \tag{B.4}
\end{align*}
$$

From Lemma 3.4, we have

$$
\begin{aligned}
\left\langle\nabla f\left(\boldsymbol{x}^{k}\right)\right. & \left.-\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \bar{x}^{k+1}-x^{k}\right\rangle \mathrm{v} \\
& \leq \tau L_{\mathrm{res}}^{\vee} \sqrt{\mathrm{p}_{\max }} \sum_{i=0}^{m} \mathrm{p}_{i}\left|\bar{x}_{i}^{k+1}-x_{i}^{k}\right|^{2}+\alpha_{k}-\mathrm{E}\left[\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right]
\end{aligned}
$$

with $\alpha_{k}=\frac{L_{\mathrm{res}}^{\mathrm{V}}}{2 \sqrt{\mathrm{p}_{\max }}} \sum_{h=k-\tau}^{k-1}(h-(k-\tau)+1)\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|_{\mathrm{V}}^{2}$. We then plug this result in (B.4) obtaining

$$
\begin{aligned}
& \sum_{i=0}^{m} \mathrm{p}_{i}\left(\frac{1}{\gamma_{i}}-\frac{L_{i}}{2}\right)\left|\bar{x}_{i}^{k+1}-x_{i}^{k}\right|^{2} \leq F\left(x^{k}\right)+\alpha_{k}+\tau L_{\mathrm{res}}^{\vee} \sqrt{\mathrm{p}_{\max }} \sum_{i=0}^{m} \mathrm{p}_{i}\left|\bar{x}_{i}^{k+1}-x_{i}^{k}\right|^{2} \\
&-\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right]
\end{aligned}
$$

Hence

$$
\sum_{i=0}^{m} \mathrm{p}_{i}\left(\frac{1}{\gamma_{i}}-\frac{L_{i}}{2}-\tau L_{\mathrm{res}}^{\vee} \sqrt{\mathrm{p}_{\max }}\right)\left|\bar{x}_{i}^{k+1}-x_{i}^{k}\right|^{2} \leq F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right]
$$

Since $\delta<2$, recalling (3.1), we have, for all $i \in[m]$,

$$
\left(\frac{1}{\gamma_{i}}-\frac{L_{i}}{2}-\tau L_{\mathrm{res}}^{\vee} \sqrt{\mathrm{p}_{\mathrm{max}}}\right)=\frac{1}{2 \gamma_{i}}\left(2-L_{i} \gamma_{i}-2 \gamma_{i} \tau L_{\mathrm{res}}^{\vee} \sqrt{\mathrm{p}_{\max }}\right) \geq \frac{1}{2 \gamma_{i}}(2-\delta)>0
$$

Therefore the statement follows.
Proof of Proposition 3.6. Let $k \in \mathbb{N}$ and $\mathbf{x} \in \mathbf{H}$. Since $\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \mathbf{x}-\boldsymbol{x}^{k}\right\rangle=\left\langle\nabla^{\Gamma^{-1}} f\left(\hat{\boldsymbol{x}}^{k}\right), \mathbf{x}-\boldsymbol{x}^{k}\right\rangle_{\Gamma^{-1}}$ and $\overline{\boldsymbol{x}}^{k+1}=\operatorname{prox}_{g}^{\Gamma^{-1}}\left(\boldsymbol{x}^{k}-\nabla^{\Gamma^{-1}} f\left(\hat{\boldsymbol{x}}^{k}\right)\right)$, we derive from Lemma 2.8 above written in weighted norm that

$$
\begin{align*}
\left\langle\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}, \mathbf{x}-\boldsymbol{x}^{k}\right\rangle_{\Gamma^{-1}} \leq & g(\mathbf{x})-g\left(\boldsymbol{x}^{k}\right)+\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \mathbf{x}-\boldsymbol{x}^{k}\right\rangle \\
& +g\left(\boldsymbol{x}^{k}\right)-g\left(\overline{\boldsymbol{x}}^{k+1}\right)+\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\rangle \\
& -\left\|\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\|_{\Gamma^{-1}}^{2} \tag{B.5}
\end{align*}
$$

From Lemma 2.7, we have

$$
\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \mathbf{x}-\boldsymbol{x}^{k}\right\rangle \leq f(\mathbf{x})-f\left(\boldsymbol{x}^{k}\right)+\frac{\tau L_{\mathrm{res}}}{2} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2}
$$

So (B.5) becomes

$$
\begin{align*}
\left\langle\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}, \mathbf{x}-\boldsymbol{x}^{k}\right\rangle_{\Gamma^{-1}} \leq F & (\mathbf{x})-F\left(\boldsymbol{x}^{k}\right)+\frac{\tau L_{\mathrm{res}}}{2} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2} \\
& +g\left(\boldsymbol{x}^{k}\right)-g\left(\overline{\boldsymbol{x}}^{k+1}\right)+\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\rangle \\
& -\left\|\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\|_{\Gamma-1}^{2} \tag{B.6}
\end{align*}
$$

Next, recalling that $x^{k}$ and $x^{k+1}$ differs only in the $i_{k}$-th component, we have

$$
\begin{aligned}
g\left(\boldsymbol{x}^{k}\right)-g\left(\overline{\boldsymbol{x}}^{k+1}\right) & +\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\rangle \\
& =\mathrm{E}\left[\left.\sum_{i=1}^{m} \frac{1}{\mathrm{p}_{i}}\left(g_{i}\left(x_{i}^{k}\right)-g_{i}\left(x_{i}^{k+1}\right)+\left\langle\nabla_{i} f\left(\hat{\boldsymbol{x}}^{k}\right), x_{i}^{k}-x_{i}^{k+1}\right\rangle\right) \right\rvert\, i_{0}, \ldots, i_{k-1}\right]
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
& \sum_{i=1}^{m} \frac{1}{\mathbf{p}_{i}}\left(g_{i}\left(x_{i}^{k}\right)-g_{i}\left(x_{i}^{k+1}\right)+\left\langle\nabla_{i} f\left(\hat{\boldsymbol{x}}^{k}\right), x_{i}^{k}-x_{i}^{k+1}\right\rangle\right) \\
&=\frac{1}{\mathbf{p}_{\text {min }}}\left(g\left(\boldsymbol{x}^{k}\right)-g\left(\boldsymbol{x}^{k+1}\right)+\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \boldsymbol{x}^{k}-\boldsymbol{x}^{k+1}\right\rangle\right) \\
& \quad-\sum_{i=1}^{m}(\underbrace{\left.\frac{1}{\mathbf{p}_{\text {min }}}-\frac{1}{\mathbf{p}_{i}}\right)\left(g_{i}\left(x_{i}^{k}\right)-g_{i}\left(x_{i}^{k+1}\right)+\left\langle\nabla_{i} f\left(\hat{\boldsymbol{x}}^{k}\right), x_{i}^{k}-x_{i}^{k+1}\right\rangle\right)}_{\geq 0} \\
& \quad \leq \frac{1}{\mathbf{p}_{\text {min }}}\left(g\left(\boldsymbol{x}^{k}\right)-g\left(\boldsymbol{x}^{k+1}\right)+\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \boldsymbol{x}^{k}-\boldsymbol{x}^{k+1}\right\rangle\right) \\
& \quad-\left(\frac{1}{\mathbf{p}_{\text {min }}}-\frac{1}{\mathbf{p}_{i_{k}}}\right) \frac{1}{\gamma_{i_{k}}}\left|\Delta_{i_{k}}^{k}\right|^{2}
\end{aligned}
$$

where in the last inequality we used that

$$
-\left(g_{i_{k}}\left(x_{i_{k}}^{k}\right)-g_{i_{k}}\left(x_{i_{k}}^{k+1}\right)+\left\langle\nabla_{i_{k}} f\left(\hat{\boldsymbol{x}}^{k}\right), x_{i_{k}}^{k}-x_{i_{k}}^{k+1}\right\rangle\right) \leq-\frac{1}{\gamma_{i_{k}}}\left|\Delta_{i_{k}}^{k}\right|^{2},
$$

which was derived from (2.5). So

$$
\begin{aligned}
g\left(\boldsymbol{x}^{k}\right)-g\left(\overline{\boldsymbol{x}}^{k+1}\right)+ & \left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\rangle \\
\leq & \frac{1}{\mathrm{p}_{\min }} \mathrm{E}\left[g\left(\boldsymbol{x}^{k}\right)-g\left(\boldsymbol{x}^{k+1}\right)+\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \boldsymbol{x}^{k}-\boldsymbol{x}^{k+1}\right\rangle \mid i_{0}, \ldots, i_{k-1}\right] \\
& -\frac{1}{\mathrm{p}_{\min }} \sum_{i=1}^{m} \frac{\mathrm{p}_{i}}{\gamma_{i}}\left|\Delta_{i}^{k}\right|^{2}+\left\|\boldsymbol{x}^{k}-\boldsymbol{x}^{k+1}\right\|_{\Gamma-1}^{2} .
\end{aligned}
$$

Now, by Lemma 3.4 and the block-coordinate descent lemma (2.6), we have

$$
\begin{aligned}
& \mathrm{E}\left[\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \boldsymbol{x}^{k}-\boldsymbol{x}^{k+1}\right\rangle \mid i_{0}, \ldots, i_{k-1}\right] \\
& \leq \mathrm{E}\left[\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right)-\nabla f\left(\boldsymbol{x}^{k}\right), \boldsymbol{x}^{k}-\boldsymbol{x}^{k+1}\right\rangle \mid i_{0}, \ldots, i_{k-1}\right]+\mathrm{E}\left[\left\langle\nabla f\left(\boldsymbol{x}^{k}\right), \boldsymbol{x}^{k}-\boldsymbol{x}^{k+1}\right\rangle \mid i_{0}, \ldots, i_{k-1}\right] \\
&=\left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right)-\nabla f\left(\boldsymbol{x}^{k}\right), \boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\rangle \mathrm{V}+\mathrm{E}\left[\left\langle\nabla f\left(\boldsymbol{x}^{k}\right), \boldsymbol{x}^{k}-\boldsymbol{x}^{k+1}\right\rangle \mid i_{0}, \ldots, i_{k-1}\right] \\
& \leq \tau L_{\mathrm{res}}^{\vee} \sqrt{\mathrm{P}_{\max }} \sum_{i=0}^{m} \mathrm{p}_{i}\left|\bar{x}_{i}^{k+1}-x_{i}^{k}\right|^{2}+\alpha_{k}-\mathrm{E}\left[\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right] \\
& \quad+\mathrm{E}\left[\left.f\left(\boldsymbol{x}^{k}\right)-f\left(\boldsymbol{x}^{k+1}\right)+\frac{L_{i_{k}}}{2}\left|\Delta_{i_{k}}^{k}\right|^{2} \right\rvert\, i_{0}, \ldots, i_{k-1}\right],
\end{aligned}
$$

where $\alpha_{k}=L_{\mathrm{res}}^{\vee} /\left(2 \sqrt{\mathrm{P}_{\text {max }}}\right) \sum_{h=k-\tau}^{k-1}(h-(k-\tau)+1)\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|_{\mathrm{V}}^{2}$ for all $k \in \mathbb{N}$. Therefore

$$
\begin{align*}
g\left(\boldsymbol{x}^{k}\right)-g\left(\overline{\boldsymbol{x}}^{k+1}\right)+ & \left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\rangle \\
\leq & \frac{1}{\boldsymbol{p}_{\min }} \mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-F\left(x^{k+1}\right)-\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right] \\
& +\frac{1}{\mathbf{p}_{\min }} \sum_{i=1}^{m} \mathrm{p}_{i}\left(\frac{L_{i}}{2}+\tau L_{\mathrm{res}}^{\vee} \sqrt{\boldsymbol{p}_{\max }}-\frac{1}{\gamma_{i}}\right)\left|\Delta_{i}^{k}\right|^{2}+\left\|\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\|_{\mathrm{r}-1}^{2} . \tag{B.7}
\end{align*}
$$

Since $\gamma_{i} L_{i}+2 \gamma_{i} \tau L_{\text {res }}^{V} \sqrt{p_{\text {max }}} \leq \delta<2$, we have

$$
\frac{L_{i}}{2}+\tau L_{\mathrm{res}}^{\vee} \sqrt{\boldsymbol{p}_{\max }}-\frac{1}{\gamma_{i}}=\frac{1}{2 \gamma_{i}}\left(\gamma_{i} L_{i}+2 \gamma_{i} \tau L_{\mathrm{res}}^{\vee} \sqrt{\mathrm{p}_{\max }}-2\right)<0
$$

and hence (B.7) yields

$$
\begin{aligned}
g\left(\boldsymbol{x}^{k}\right)-g\left(\overline{\boldsymbol{x}}^{k+1}\right)+ & \left\langle\nabla f\left(\hat{\boldsymbol{x}}^{k}\right), \boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\rangle \\
\leq & \frac{1}{\mathrm{p}_{\min }} \mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-F\left(x^{k+1}\right)-\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right] \\
& +\frac{\delta-2}{2} \sum_{i=1}^{m} \frac{1}{\gamma_{i}}\left|\Delta_{i}^{k}\right|^{2}+\left\|\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\|_{\Gamma^{-1}}^{2} .
\end{aligned}
$$

The statement follows from (B.6).
Proof of Proposition 3.7. We know that

$$
\left\|\boldsymbol{x}^{k}-\mathbf{x}\right\|_{\Gamma^{-1}}^{2}-\left\|\overline{\boldsymbol{x}}^{k+1}-\mathbf{x}\right\|_{\Gamma^{-1}}^{2}=-\left\|\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\|_{\Gamma^{-1}}^{2}+2\left\langle\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}, \boldsymbol{x}^{k}-\mathbf{x}\right\rangle_{\Gamma^{-1}}
$$

We derive from Proposition 3.6, multiplied by 2, that

$$
\begin{align*}
\left\|\overline{\boldsymbol{x}}^{k+1}-\mathbf{x}\right\|_{\Gamma^{-1}}^{2} \leq & \left\|\boldsymbol{x}^{k}-\mathbf{x}\right\|_{\Gamma-1}^{2} \\
& +\frac{2}{\mathrm{p}_{\min }} \mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-F\left(\boldsymbol{x}^{k+1}\right)-\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right] \\
& +2\left(F(\mathbf{x})-F\left(\boldsymbol{x}^{k}\right)\right)+\tau L_{\mathrm{res}} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2} \\
& (\delta-1)\left\|\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\|_{\Gamma-1}^{2} . \tag{B.8}
\end{align*}
$$

where $\alpha_{k}=L_{\mathrm{res}}^{\vee} /\left(2 \sqrt{\mathrm{P}_{\max }}\right) \sum_{h=k-\tau}^{k-1}(h-(k-\tau)+1)\left\|\boldsymbol{x}^{h+1}-\boldsymbol{x}^{h}\right\|_{\mathrm{V}}^{2}$. It follows from Lemma 3.5 that

$$
\begin{align*}
\mathrm{E}\left[\left\|\boldsymbol{x}^{k+1}-\mathbf{x}\right\|_{\mathrm{W}}^{2} \mid\right. & \left.i_{0}, \ldots, i_{k-1}\right] \\
\leq & \left\|\boldsymbol{x}^{k}-\mathbf{x}\right\|_{\mathrm{W}}^{2} \\
& +(\delta-1)\left\|\boldsymbol{x}^{k}-\overline{\boldsymbol{x}}^{k+1}\right\|_{\Gamma^{-1}}^{2} \\
& +\frac{2}{\mathrm{p}_{\min }} \mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-F\left(\boldsymbol{x}^{k+1}\right)-\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right] \\
& +2\left(F(\mathbf{x})-F\left(\boldsymbol{x}^{k}\right)\right)+\tau L_{\mathrm{res}} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2} . \tag{B.9}
\end{align*}
$$

Plugging (3.2) in (B.9) the statement follows.
Proof of Proposition 3.8. Let $k \in \mathbb{N}$ and $\mathbf{x} \in \mathbf{H}$. From Proposition 3.7, we have

$$
\begin{aligned}
& \mathrm{E}\left[\left\|\boldsymbol{x}^{k+1}-\mathbf{x}\right\|_{\mathrm{W}}^{2} \mid i_{0}, \ldots, i_{k-1}\right] \\
& \leq\left\|\boldsymbol{x}^{k}-\mathbf{x}\right\|_{\mathrm{W}}^{2} \\
&+\frac{2}{\mathrm{p}_{\text {min }}}\left(\frac{(\delta-1)_{+}}{2-\delta}+1\right) \mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-F\left(\boldsymbol{x}^{k+1}\right)-\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right] \\
&+\tau L_{\text {res }} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2} \\
&+2\left(F(\mathbf{x})-\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right]\right) . \\
&-2\left(\mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-F\left(\boldsymbol{x}^{k+1}\right)-\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right]\right)+2 \alpha_{k}
\end{aligned}
$$

Set for all $k \in \mathbb{N}$,

$$
\begin{aligned}
\xi_{k}= & 2\left(\frac{\max \left\{1,(2-\delta)^{-1}\right\}}{\mathrm{p}_{\min }}-1\right) \mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-F\left(\boldsymbol{x}^{k+1}\right)-\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right] \\
& +\tau L_{\mathrm{res}} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2}+2 \alpha_{k} .
\end{aligned}
$$

Now, on the one hand, recalling (B.14), (3.2) and Lemma 3.5, we have

$$
\begin{aligned}
\mathrm{E}\left[\sum_{k \in \mathbb{N}} \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2}\right] & \leq \tau \gamma_{\max } \mathrm{p}_{\max } \sum_{k \in \mathbb{N}} \mathrm{E}\left[\left\|\boldsymbol{x}^{k}-\boldsymbol{x}^{k+1}\right\|_{\mathrm{W}}^{2}\right] \\
& \leq \frac{2 \tau \gamma_{\max } \mathrm{p}_{\max }}{(2-\delta) \mathrm{p}_{\min }} \sum_{k \in \mathbb{N}}\left(\mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}\right]-\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)-\alpha_{k+1}\right]\right) \\
& \leq \frac{2 \tau \gamma_{\max } \mathrm{p}_{\max }}{(2-\delta) \mathrm{p}_{\min }}\left(F\left(\boldsymbol{x}^{0}\right)+\alpha_{0}-F^{*}\right)<+\infty
\end{aligned}
$$

Recalling the definition of $\alpha_{k}$ in Proposition 3.6 and of $L_{\text {res }}^{\vee}$ in Remark 2.6, this also yields

$$
\begin{aligned}
\mathrm{E}\left[\sum_{k \in \mathbb{N}} \alpha_{k}\right] & \leq \frac{\tau L_{\mathrm{res}}^{\vee}}{2 \sqrt{\mathrm{p}_{\max }}} \mathrm{E}\left[\sum_{k \in \mathbb{N}} \sum_{h=k-\tau}^{k-1}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|_{\mathrm{V}}^{2}\right] \\
& \leq \frac{\tau L_{\mathrm{res}}^{\vee} \mathrm{p}_{\max }}{2 \sqrt{\mathrm{p}_{\max }}} \mathrm{E}\left[\sum_{k \in \mathbb{N}} \sum_{h=k-\tau}^{k-1}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2}\right] \\
& \leq \frac{\tau L_{\mathrm{res}} \mathrm{p}_{\max }^{2}}{\sqrt{\mathrm{p}_{\min }}} \frac{\tau \gamma_{\max }}{(2-\delta) \mathrm{p}_{\min }}\left(F\left(\boldsymbol{x}^{0}\right)+\alpha_{0}-F^{*}\right) .
\end{aligned}
$$

On the other hand, setting $\eta_{k}=F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}-\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)-\alpha_{k+1} \mid i_{0}, \ldots, i_{k-1}\right]$, which in virtue of (3.2) is positive P -a.s., we have

$$
\mathrm{E}\left[\sum_{k \in \mathbb{N}} \eta_{k}\right]=\sum_{k \in \mathbb{N}} \mathrm{E}\left[\eta_{k}\right]=\sup _{n \in \mathbb{N}} \sum_{k=0}^{n} \mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}\right]-\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)-\alpha_{k+1}\right] \leq F\left(\boldsymbol{x}^{0}\right)+\alpha_{0}-F^{*}<+\infty .
$$

Let $C=\frac{\max \left\{1,(2-\delta)^{-1}\right\}}{\mathrm{p}_{\min }}-1+\tau^{2} \frac{L_{\mathrm{res}} \gamma_{\max } \mathrm{p}_{\max }}{\mathrm{p}_{\min }(2-\delta)}\left(1+\frac{\mathrm{p}_{\max }}{\sqrt{\mathrm{p}_{\min }}}\right)$. We then get

$$
\sum_{k \in \mathbb{N}} \mathrm{E}\left[\xi_{k}\right] \leq 2 C\left(F\left(\boldsymbol{x}^{0}\right)-F^{*}\right)
$$

We remark that $(\forall i \in[m]) \quad \gamma_{i}\left(L_{i}+2 \tau L_{\mathrm{res}} \mathrm{p}_{\max } / \sqrt{\mathrm{p}_{\min }}\right)<2$. So $\gamma_{i} \tau L_{\mathrm{res}}<\frac{2-\gamma_{i} L_{i}}{2} \frac{\sqrt{\mathrm{p}_{\text {min }}}}{\mathrm{p}_{\max }}$. This implies $\tau \gamma_{\text {max }} L_{\text {res }}<\frac{2-\gamma_{\max } L_{i_{0}}}{2} \frac{\sqrt{\mathrm{p}_{\text {min }}}}{\mathrm{p}_{\text {max }}}$, where $i_{0} \in[\mathrm{~m}]$ such that $\gamma_{i_{0}}=\gamma_{\text {max }}$. Thus

$$
\begin{equation*}
\tau \gamma_{\max } L_{\mathrm{res}}<\frac{2-\gamma_{\max } L_{\min }}{2} \frac{\sqrt{\mathrm{p}_{\min }}}{\mathrm{p}_{\max }} \tag{B.10}
\end{equation*}
$$

Using this in $C$, we get

$$
\begin{aligned}
C & \leq \frac{\max \left\{1,(2-\delta)^{-1}\right\}}{\mathrm{p}_{\min }}-1+\tau \frac{2-\gamma_{\max } L_{\min }}{2 \sqrt{\mathrm{p}_{\min }}(2-\delta)}\left(1+\frac{\mathrm{p}_{\max }}{\sqrt{\mathrm{p}_{\min }}}\right) \\
& \leq \frac{\max \left\{1,(2-\delta)^{-1}\right\}}{\mathrm{p}_{\min }}-1+\tau \frac{1}{\sqrt{\mathrm{p}_{\min }}(2-\delta)}\left(1+\frac{\mathrm{p}_{\max }}{\sqrt{\mathrm{p}_{\min }}}\right) .
\end{aligned}
$$

The statement follows.
Proof of Proposition 3.9. It follows from (3.2) that

$$
(2-\delta) \frac{\mathrm{p}_{\min }}{2} \mathrm{E}\left[\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\|_{\Gamma_{-1}}^{2}\right] \leq \mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}\right]-\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1}\right]
$$

This means that $\left(\mathrm{E}\left[F\left(\boldsymbol{x}^{k}\right)+\alpha_{k}\right]\right)_{k \in \mathbb{N}}$ is a nonincreasing sequence and

$$
\begin{aligned}
(2-\delta) \frac{\mathrm{p}_{\text {min }}}{2} \mathrm{E}\left[\sum_{k \in \mathbb{N}}\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\|_{\Gamma^{-1}}^{2}\right] & =(2-\delta) \frac{\mathrm{p}_{\min }}{2} \sup _{k \in \mathbb{N}} \sum_{h=0}^{k} \mathrm{E}\left[\left\|\overline{\boldsymbol{x}}^{h+1}-\boldsymbol{x}^{h}\right\|_{\Gamma^{-1}}^{2}\right] \\
& \leq \sup _{k \in \mathbb{N}} \mathrm{E}\left[F\left(\boldsymbol{x}^{0}\right)+\alpha_{0}\right]-\mathrm{E}\left[F\left(\boldsymbol{x}^{k+1}\right)+\alpha_{k+1}\right] \\
& \leq F\left(\boldsymbol{x}^{0}\right)+\alpha_{0}-F^{*}<+\infty .
\end{aligned}
$$

Therefore, since $\|\cdot\|^{2} \leq\left(\max _{i} \gamma_{i}\right)\|\cdot\|_{\Gamma-1}^{2}$, we derive that

$$
\begin{equation*}
\sum_{k \in \mathbb{N}}\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\|^{2}<\infty \quad \text { P-a.s. } \tag{B.11}
\end{equation*}
$$

So, it follows that

$$
\begin{equation*}
\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\| \rightarrow 0 \quad \text { P-a.s, } \tag{B.12}
\end{equation*}
$$

and, since $\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\| \leq\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\|$ for all $k \in \mathbb{N}$, we have also

$$
\begin{equation*}
\sum_{k \in \mathbb{N}}\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\|^{2}<\infty \text { and }\left\|\boldsymbol{x}^{k+1}-\boldsymbol{x}^{k}\right\| \rightarrow 0 \quad \text { P-a.s. } \tag{B.13}
\end{equation*}
$$

Now, by Lemma 2.4, we have $\left\|\hat{\boldsymbol{x}}^{k}-\boldsymbol{x}^{k}\right\|^{2} \leq \tau \sum_{h \in J(k)}\left\|\boldsymbol{x}^{h}-\boldsymbol{x}^{h+1}\right\|^{2}$ and, moreover,

$$
\begin{equation*}
\sum_{k \in \mathbb{N}} \sum_{h \in J(k)}\left\|x^{h}-x^{h+1}\right\|^{2} \leq \sum_{k \in \mathbb{N}} \tau\left\|x^{k}-\boldsymbol{x}^{k+1}\right\|^{2}<\infty \quad \text { P-a.s. }, \tag{B.14}
\end{equation*}
$$

so that

$$
\begin{equation*}
\left\|\overline{\boldsymbol{x}}^{k+1}-\hat{\boldsymbol{x}}^{k}\right\| \leq\left\|\overline{\boldsymbol{x}}^{k+1}-\boldsymbol{x}^{k}\right\|+\left\|\boldsymbol{x}^{k}-\hat{\boldsymbol{x}}^{k}\right\| \rightarrow 0 \text { P-a.s. } \tag{B.15}
\end{equation*}
$$

Define, for all $i \in[m]$,

$$
\begin{equation*}
v_{i}^{k}=\nabla_{i} f\left(\overline{\boldsymbol{x}}^{k+1}\right)-\nabla_{i} f\left(\hat{\boldsymbol{x}}^{k}\right)+\frac{\Delta_{i}^{k}}{\gamma_{i}} . \tag{B.16}
\end{equation*}
$$

Then, thanks to the second equation in (2.4), we have

$$
\begin{equation*}
\boldsymbol{v}^{k}=\left(v_{1}^{k}, \cdots, v_{m}^{k}\right) \in \nabla f\left(\overline{\boldsymbol{x}}^{k+1}\right)+\partial g\left(\overline{\boldsymbol{x}}^{k+1}\right)=\partial(f+g)\left(\overline{\boldsymbol{x}}^{k+1}\right) . \tag{B.17}
\end{equation*}
$$

Moreover, since $\nabla f$ is Lipschitz continuous, definition (B.16) and equations (B.12), (B.15) yield $\boldsymbol{v}^{k} \rightarrow 0$ P-a.s.

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[^0]:    *Malga Center, DIMA, Università degli Studi di Genova, Genoa, Italy (traore@dima.unige.it).
    "Istituto Italiano di Tecnologia, Genova, Italy (saverio.salzo@iit.it).
    ${ }^{*}$ Malga Center, DIMA, Università degli Studi di Genova, Genova, Italy (silvia.villa@unige.it).

