A gauge-invariant reversible cellular automaton

Pablo Arrighi[1], Giuseppe Di Molfetta[2], and Nathanaël Eon[3]

[1] Aix-Marseille Univ, Université de Toulon, CNRS, LIS, Marseille, France and IXXI, Lyon, France
[2] Aix-Marseille Univ, Université de Toulon, CNRS, LIS, Marseille, France and Departamento de Física Terica and IFIC, Universidad de Valencia-CSIC, Dr. Moliner 50, 46100-Burjassot, Spain
[3] Aix Marseille Univ, Universite de Toulon, CNRS, LIS, Marseille, France
École Centrale

Abstract. Gauge-invariance is a fundamental concept in physics—known to provide the mathematical justification for all four fundamental forces. In this paper, we provide discrete counterparts to the main gauge theoretical concepts, directly in terms of Cellular Automata. More precisely, we describe a step-by-step gauging procedure to enforce local symmetries upon a given Cellular Automaton. We apply it to a simple Reversible Cellular Automaton for concreteness. From a Computer Science perspective, discretized gauge theories may be applied to numerical analysis, quantum simulation, fault-tolerant (quantum) computation. From a mathematical perspective, discreteness provides a simple yet rigorous route straight to the core concepts.

Keywords: reversible cellular automata, gauge theory, error correction

1 Introduction

In Physics, symmetries are our guiding principles towards discovering and modeling the laws we put forward to model nature. Among them, Gauge symmetries are absolutely central, as they provide the mathematical justification for all four fundamental forces: electromagnetism and gravity (long range interactions), weak and strong forces (short range interactions) [1]. In this paper we express the key notions of gauge theories natively in Computer Science friendly, Discrete Mathematics terms—we do so in order to make them available to the discipline, and in order to clarify its concepts. More precisely, we describe a discrete counterpart to the gauging procedure. I.e. we thereby provide a step-by-step procedure to enforce local symmetries within Cellular Automata.

These methods may lead to natural, physics-inspired CA. The fields of numerical analysis, quantum simulation, digital physics are constantly looking for discrete schemes that simulate known physics [2]. Quite often, these discrete schemes seek to retain the symmetries of the simulated physics; whether in order to justify the discrete scheme as legitimate, or in order to do the Monte Carlo-counting right [3]. Since gauge symmetries are essential in physics, having a discrete counterpart of it may also be.

This way of enforcing local redundancies also bears some resemblances with error-correction, and echoes the fascinating question of noise resistance within spatially–distributed models of computation [4,5], as was pointed out in the context of quantum computation in [6,7].

Although we authors come from the field of quantum computation and simulation, the formalism we use is totally devoid of any quantum theory, least action principle and Lagrangian. The notions are directly formulated in terms of the discrete dynamical system. We believe that this provides a uniquely direct route to the root concepts. This discrete mathematics framework makes the presentation original, and simpler. But it also allows for more rigorous definitions, that in turn allow us to prove some essential consistency lemmas that are usually left aside. Our running example provides what seems to be the simplest non-trivial Gauge theory so far and illustrates the key concepts. Given the fame of Gauge theories, we think this may be a remarkable pedagogical and unexplored asset.

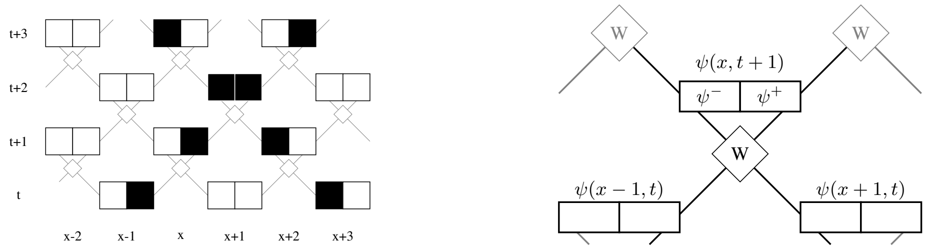
The paper is organized as follows. In Sec. 2 we introduce the notions of local transformations which define the desired symmetry, and of gauge-invariance which captures the (non-)compliance of a given Cellular Automaton (CA) with the desired symmetry. In Sec. 3 we show how a non-gauge-invariant CA can be made gauge-invariant, at the heavy cost of becoming spacetime dependent upon an external parameter, referred to as the gauge field. This new parameter not only implements the symmetry—it leads to new behaviours for the CA. In Sec. 4 the gauge field gets internalized into the configuration space, and a whole family of homogeneous gauge-invariant CA is obtained, leading us to the notions of gauge-fixing and gauge-constraining. A simple Reversible Cellular Automaton (RCA) is used to illustrate each concepts, throughout the paper. In Sec. 5 we summarize, provide related works and perspectives.

2 The gauge-invariance requirements

Theory to be gauged. In this paper 'theories' stand for CA. As our running example, we pick possibly the simplest and most natural physics-like RCA : one that has particles moving left and right. More precisely, each cell of the RCA has a state in $\Sigma = \{\square\square, \square\blacksquare, \blacksquare\square, \blacksquare\blacksquare\} \cong \{00, 01, 10, 11\}$. Its dynamics is defined through a local rule $R : \Sigma^2 \to \Sigma$ which computes the next state of a cell from that of its left and right neighbours, i.e. $\psi(x, t+1) = R(\psi(x-1, t), \psi(x+1, t))$, with $\psi(x, t)$ the state of cell x at time t. A spacetime diagram $\psi : \mathbb{Z}^2 \to \Sigma$, is said to be R–valid iff produced by applying R, see for instance Fig-1a. The R that we consider can be expressed in the block circuit form of Fig-1b, with

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

RCA presented in such a block circuit form are often referred to as (Margolus-)Partitioned CA in Computer Science vocabulary[8], or as Lattice-gas automata in Physics[9]. This theory is *to-be-gauged*. This means that although it may have a global symmetry (here the CA has global black/white–symmetry, see Fig-2 $(a) - (b)$), it lacks a certain local symmetry (here no deterministic CA describes Fig-2 (c)). The aim of the so-called *gauging procedure* is to extend a theory order so as to enforce a given local symmetry.



(a) A valid space-time diagram with two particles moving in opposite directions

(b) Framework of study

Fig. 1: Representation of the framework of study.

Local transformation and its invariant. In our running example we will be interested in enforcing a *local* black/white–symmetry. We formalize this by giving ourselves a bit field $\varphi : \mathbb{Z}^2 \to \{0,1\} \cong \mathbb{Z}_2$ that specifies, at each spacetime point, whether the symmetry is to be applied. In other words, the action of the $\mathbb{Z}_2$ group gets represented upon Σ by

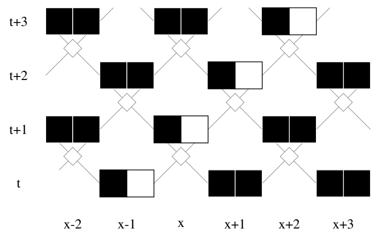
$$G_\varphi = X^\varphi \otimes X^\varphi \quad \text{with} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, an entire spacetime diagram ψ transforms into an entire spacetime diagram $G_\varphi\psi$ via $(G_\varphi\psi)(x,t) = G_{\varphi(x,t)}\psi(x,t)$. This is the symmetry we are trying to enforce. Thus, whenever two spacetime diagrams are related by a transformation G_φ for some φ, they are understood as *physically equivalent*. For instance, in Fig-2, the first three diagrams are physically equivalent with respect to the symmetry—even the third one. They all represent this one particle moving right, which can be understood as an invariant of the symmetry. Given a spacetime diagram ψ, we write $\widetilde{\psi} = \{G_\varphi\psi \,|\, \varphi \in \mathbb{Z}^2 \to \mathbb{Z}_2\}$ for its *invariant*, (physical) equivalence class. In the case of our field $\psi = (\psi^-, \psi^+)^T$ (as shown in Fig-1b) we introduce the field $J(x,t) = \psi^+(x,t) - \psi^-(x,t)$ which is our invariant and fully characterizes $\widetilde{\psi}$, since for all ψ and ψ', $G_\varphi\psi = \psi'$ if and only if $J = J'$. Fig-2d shows the underlying J.

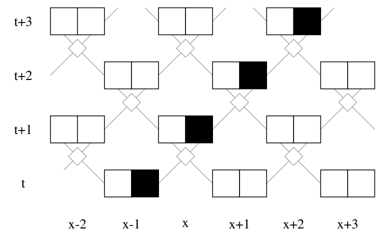
The gauge invariance condition. Given $\psi(.,t)$ and $G_{\varphi(.,t)}\psi(.,t)$ two physically equivalent inputs, it should be the case that our theory produces two physically equivalent outputs $\psi(.,t+1)$ and $G_{\varphi(.,t+1)}\psi(.,t+1)$. Generally speaking, for this to happen, a local rule T has to verify the following condition :

$$\begin{aligned} &\forall\varphi(x-1,t), \varphi(x+1,t), \exists\varphi(x,t+1), \\ &\psi(x,t+1) = T(\psi(x-1,t), \psi(x+1,t)) \\ &\;\Rightarrow G_{\varphi(x,t+1)}\psi(x,t+1) = T(G_{\varphi(x-1,t)}\psi(x-1,t), G_{\varphi(x+1,t)}\psi(x+1,t)) \end{aligned} \tag{1}$$

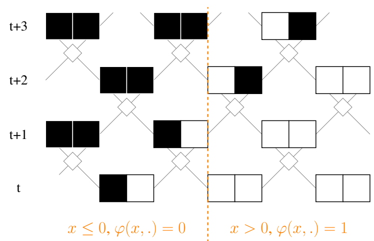
This is the the *gauge-invariance* condition. The above-defined RCA fails to meet this requirement. An example of this failure is provided by Fig-2, which shows three physically equivalent spacetime diagrams, i.e. that are G_φ–related. Clearly the first two are



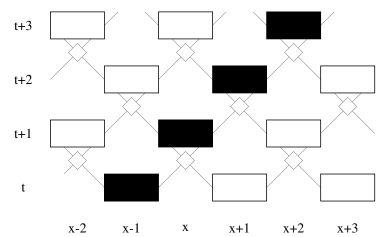
(a) R–valid spacetime diagram showing a particle moving right.



(b) Still an R–valid spacetime diagram after applying the global symmetry G_φ with φ constant equal to one.



(c) Not an R–valid spacetime diagram after applying the local symmetry G_φ with space-dependent φ.



(d) The J-field that characterizes the invariant under G_φ, common to the other three spacetime diagrams.

Fig. 2: Three physically equivalent spacetime diagrams, and their invariant.

R–valid, but the third one is not, as can be seen from looking at $\psi(x, t+1)$. The important point is that this cannot be fixed with a better choice of $\varphi(x, t+1)$.
Indeed, on the one hand cell $\psi(x, t+1)$ of Fig-2c needs have different-color subcells, as it is a $G_{\varphi(x,t+1)}$ of that of the other diagrams, and $G_{\varphi(x,t+1)}$ conserves same-colorness. But, on the other hand, cell $\psi(x, t+1)$ of Fig-2c needs have same-color subcells, as it is produced by a W which is fed with same-color subcells—due to the particular choice of $\varphi(x-1, t)$ and $\varphi(x+1, t)$—and W conserves same-colorness. Therefore, our previously defined RCA fails to verify the gauge-invariance condition.
The gauging procedure proceeds by extending R into an inhomogeneous dynamics.

3 The gauge field

Introducing the gauge field. In order to obtain the gauge-invariance condition (1), we introduce the *gauge field*, namely in mathematical language the Ehresmann connection, $A(x, t)$ and make the local rule R into an $A(x, t)$–dependent rule, which we denote by $R_\bullet$. Now, analyzing the dynamics, the gauge field $A(x, t)$ must be treated as a dynamical field, similar to other objects in the description of a physical situation. However, for $R_\bullet$ to verify gauge-invariance, we will need to transform A at the same time as we transform ψ—otherwise we will again run into problems in the style of those encountered in Fig-2. Therefore we extend G_φ to act on both ψ and A, at the same time as we aim for condition (1). Developing the condition, the now A–dependent $W_\bullet$ of $R_\bullet$ must verify :

$$
\begin{aligned}
&\forall A, \varphi(x-1,t), \varphi(x+1,t), \exists\varphi(x,t+1),\\
&\psi(x,t+1) = W_A \begin{pmatrix}\psi^+(x-1,t)\\ \psi^-(x+1,t)\end{pmatrix}\\
&\Rightarrow (X^{\varphi(x,t+1)} \otimes X^{\varphi(x,t+1)})\psi(x,t+1) = W_{G_\varphi A} \begin{pmatrix}X^{\varphi(x-1,t)}\psi^+(x-1,t)\\ X^{\varphi(x+1,t)}\psi^-(x+1,t)\end{pmatrix}
\end{aligned}
$$

with $\psi(x,t) = \begin{pmatrix}\psi^-(x,t)\\ \psi^+(x,t)\end{pmatrix}$. This is equivalent to

$$W_{G_\varphi A} = (X^{\varphi(x,t+1)} \otimes X^{\varphi(x,t+1)}) W_A (X^{-\varphi(x-1,t)} \otimes X^{-\varphi(x+1,t)}).$$

A somewhat minimal choice verifying the above condition is to take $A : \mathbb{Z}^2 \to \mathbb{Z}_2^2$ a 2-bit field, and

$$W_A = W(X^{A_r} \otimes X^{A_l}),$$

with A transforming under G_φ as :

$$A = \begin{pmatrix}A_r\\ A_l\end{pmatrix} \mapsto \begin{pmatrix}A_r + \varphi(x,t+1) - \varphi(x-1,t)\\ A_l + \varphi(x,t+1) - \varphi(x+1,t)\end{pmatrix} = G_\varphi A$$

Indeed,

$$
\begin{aligned}
W_{G_\varphi A} &= W(X^{A_r + \varphi(x,t+1) - \varphi(x-1,t)} \otimes X^{A_l + \varphi(x,t+1) - \varphi(x+1,t)})\\
&= W(X^{\varphi(x,t+1)} \otimes X^{\varphi(x,t+1)})(X^{A_r} \otimes X^{A_l})(X^{-\varphi(x-1,t)} \otimes X^{-\varphi(x+1,t)})\\
&= (X^{\varphi(x,t+1)} \otimes X^{\varphi(x,t+1)}) W_A (X^{-\varphi(x-1,t)} \otimes X^{-\varphi(x+1,t)}).
\end{aligned}
$$

It follows that the induced $R_\bullet$ verifies the gauge-invariance condition (1) and we say that $R_\bullet$ is the *gauge covariant* version of the gauge theory accounting the dynamical field $A(x,t)$. This procedure is reminiscent of the route physicist follow to account a local phase transformation on the state vector $\psi(x,t)$, which leads to the modern formulation of the Electrodynamics, where $A(x,t)$ play the role of the electromagnetic potential.

Invariant of the gauge field. Since A also transforms under G_φ, we may again seek to characterize its invariant $\widetilde{A} = \{G_\varphi A \,|\, \varphi \in \mathbb{Z}^2 \to \mathbb{Z}_2\}$ by means of some field F. This time, in order to do so, we introduce the light-like derivatives

$$
\begin{aligned}
\Delta_r A(x,t) &= A(x,t+1) - A(x-1,t)\\
\Delta_l A(x,t) &= A(x,t+1) - A(x+1,t)
\end{aligned}
$$

and let $F(x,t) = \Delta_r A_l(x,t) - \Delta_l A_r(x,t)$. Notice that this field is the equivalent of the electromagnetic tensor, a differential 2-form, which is the exterior derivative of the electromagnetic potential $A(x,t)$ and whose derivative leads to the Maxwell equations, and in particular to the well known Gauss Law, which deserves more attention.

A lengthy but easy computation shows that, given any A and A', $G_\varphi A = A'$ entails that $F = F'$. The converse is harder to prove, but also true. Indeed, suppose that we are given A and A' such that $F = F'$. We want to construct a φ such that $G_\varphi A = A'$, i.e. such that we have both

$$\Delta_r \varphi = A'_r - A_r \qquad \text{and} \qquad \Delta_l \varphi = A'_l - A_l. \tag{2}$$

Clearly, the requirements (2) fix the rest of φ across spacetime. Unless they conflict. This could happen every time we close up a square. Starting from $\varphi(x,t)$, say, the requirements (2) provide two prescriptions for $\varphi(x, t+2)$, namely $\varphi(x,t) + (A'_l - A_l)(x-1,t) + (A'_r - A_r)(x,t+1)$ via the left-then-right path, and $\varphi(x,t) + (A'_r - A_r)(x+1,t) + (A'_l - A_l)(x,t+1)$ via the right-then-left path. These need be equal, i.e we weed

$$(A'_l - A_l)(x,t+1) - (A'_l - A_l)(x-1,t) = (A'_r - A_r)(x,t+1) - (A'_r - A_r)(x+1,t)$$

$$\begin{aligned} \Delta_r(A'_l - A_l)(x,t) &= \Delta_l(A'_r - A_r)(x,t) \\ \Delta_r A'_l - \Delta_r A_l &= \Delta_l A'_r - \Delta_l A_r \\ \Delta_l A_r - \Delta_r A_l &= \Delta_l A'_r - \Delta_r A'_l \\ F(x,t) &= F'(x,t) \end{aligned}$$

which is our hypothesis. It follows that φ exists and so the converse holds. F fully characterizes $\widetilde{A}$.

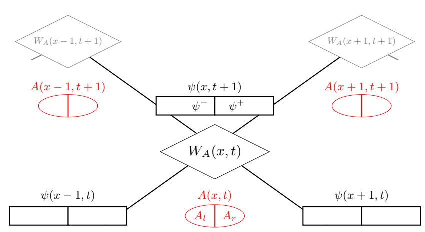
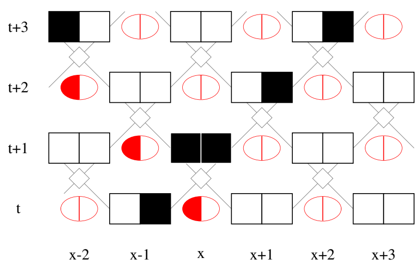


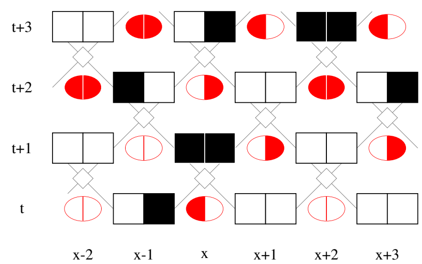
Fig. 3: Updated framework of study

Gauge field physics. It is crucial to understand that, even though A was introduced just to enforce a symmetry, i.e. to make sure that physically equivalent states are mapped into physically equivalent states. . . this newly introduced A is also capable of a range of other things, i.e. it produces new physics. For instance, Fig-4 shows how, starting from the same initial conditions but choosing different values for A as time unravels, can lead to radically different R_A-valid spacetime diagrams (sub-figures (a) and (b)) which are by no means related by a G_ϕ (e.g. in Fig-4 the configurations in the first two figures at

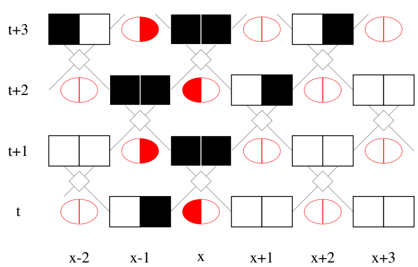
$t+2$ (and $t+3$) cannot be related by a local black/white symmetry) but also to identical ones (sub-figures (a) and (c)).
At this stage we can have two points of view upon A. Either A is seen as an independent field, which can be, to some extent at least, tuned by the user/experimentalist (in the case of the electromagnetism, we can physically control each component of the electromagnetic tensor, $F^{\alpha,\beta}(t,x)$, namely the electric and the magnetic field). Or we must extend the configuration space so as to account for A, as in Fig-3. Of course if we do that we need to provide a dynamics for A, i.e. we need to look for a local rule $c(x,t+1) = T(c(x-1,t), c(x+1,t))$, with $c(x,t) = (\psi(x,t), A(x,t))$, which still verifies the gauge-invariance condition (1).



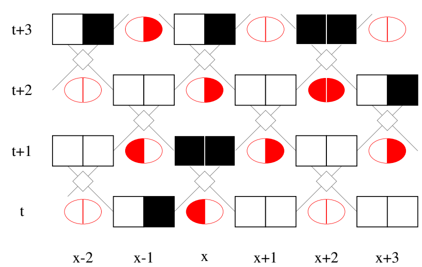
(a) A space-time diagram with a given F and $\widetilde{S}_\bullet$ where S_ψ is a swap (like our first W in section-2).



(b) A space-time diagram with a different gauge-field invariant F' leading to a non-equivalent space-time diagram. Here S_ψ is a controlled-not gate.



(c) A space-time diagram with the gauge-field invariant F (physically equivalent to (a)) and S_ψ is the identity.



(d) A space-time diagram with a different $\widetilde{S}_\bullet$, thus it is not physically equivalent to any of the other diagrams. S_ψ is a cyclic permutation.

Fig. 4: 4 space-time diagrams starting from the same initial condition and with the same rule $R_\bullet$ dictating the evolution of ψ, but with different gauge-fields. (a) and (c) have the same gauge field invariant F whereas (b) and (d) have different ones.

Gauge equivalence of two theories. We need to keep in mind that by its very nature, such a T cannot be unique—in the sense that for every candidate T there will be several

other physically equivalent local rules. This is because, as T fully implements the local symmetry, it is inherently redundant, and thus equivalent to other theories up to this redundancy. More precisely, we say that two theories T and T' are *physically equivalent theories* if and only if for any T–valid spacetime diagram c, there exists φ such that $G_\phi c$ is a T'–valid spacetime diagram.
This definition ensures that given theory T and some input configuration $c(.,t)$, we can always encode the input as $G_{\varphi(.,t)}c(.,t)$, and have it evolve under T', so as to retrieve $G_{\varphi(.,t+1)}c(.,t+1)$, which is physically equivalent to $c(.,t+1)$. We will now build candidate theories T and T' by following the standard steps of the gauging procedure.

4 Gauge field dynamics

Dynamics of its invariant. The dynamics T we want to build takes $c(.,t)$ as input and outputs $c(.,t+1)$. However, we already have $R_\bullet$ which takes $c(.,t) = (\psi(.,t), A(.,t))$ and outputs $\psi(.,t+1)$. Therefore, all we need is a rule $S_\bullet$ that will take $c(.,t)$ as input and would output $A(.,t+1)$. The standard procedure indeed proceeds by decomposing T into $W_\bullet$ and $S_\bullet$. For such a T to verify (1), we just need $S_\bullet$ to verify (1) since $W_\bullet$ already does.
It does so in two steps. The first step is to prescribe a dynamics $\widetilde{S}$ not over A, but over its invariant $\widetilde{A}$, which amounts to a dynamics over F. Or even a J–dependent dynamics $\widetilde{S}_\bullet$. Such a dynamics will always be gauge-invariant, since F and J are gauge-invariant. Thus the particular choice is only dictated by the phenomena we wish to model. A simple choice, for instance, is to take $\widetilde{S}$ constant. Then, if initially we had $F=0$, this will remain the case. Beware this does not mean that the behaviour of A is trivial. In fact it remains largely undetermined, as $F=0$, which physically transcribes into the absence of any electromagnetic field, just means $\Delta_r A_l(x,t) = \Delta_l A_r(x,t)$. But at least this constraint over the dynamics of A is gauge-invariant. In any case, this first step does not suffice to prescribe $S_\bullet$. The second step is called gauge-fixing.

Gauge-fixing : completing the dynamics. Gauge-fixing means choosing an actual $S_\bullet$ which, by meeting the gauge-invariance condition (1), induces the F and therefore the $\widetilde{S}_\bullet$ that we had settled for in the first step. More precisely, we need to fix an $S_\bullet$ such that for all (ψ, A), if A is an S_ψ–valid spacetime diagram, then F is a $\widetilde{S}_J$–valid spacetime diagram, with F and J computed from A and ψ.
This time the choice of a particular $S_\bullet$ is not guided by the physics, but by mere convenience. This is because, if two gauge-invariant $S_\bullet$ and $S'_\bullet$ induce the same F, we have the following two facts: 1/ for any A an S_ψ–valid spacetime diagram, there exists φ such that A' is an $S'_{\psi'}$–valid spacetime diagram, with $(A',\psi') = G_\varphi(A,\psi)$. 2/ the corresponding global theories $T = W_\bullet \wedge S_\bullet$ and $T' = W_\bullet \wedge S'_\bullet$, for all $W_\bullet$, are physically equivalent.

1. We will start by proving the first fact which will be useful in the proof of the second. $S_\bullet$ and $S'_\bullet$ induce the same F. Therefore, given A an $S_\bullet$-valid field and A' an $S'\bullet$-valid field, both A and A' induce the same F. Using the property, proven in the previous section, that given A and A' inducing the same F, there exists φ such that

$A' = G_\varphi A$, we prove the fact 1 by applying such a G_φ to $c = (\psi, A)$. Thus we have built a φ such that A' is an $S'_{\psi'}$–valid spacetime diagram with $(\psi', A') = G_\varphi(\psi, A)$.

2. Now we can prove fact 2. Given $c = (\psi, A)$ a T–valid spacetime diagram, consider A on its own. A is an S_ψ–valid spacetime diagram. But since $S_\bullet$ and $S'_\bullet$ both implement $\widetilde{S}_\bullet$, there exists φ such that $A' = G_\varphi A$ is an $S'_{G_\varphi \psi}$–valid spacetime diagram (fact 1). Apply this G_φ to the whole of $c = (\psi, A)$. This yields some $c' = (\psi', A')$. Is c' a T'–valid spacetime diagram? Yes, because: A' is an $S'_{\psi'}$–valid spacetime diagram by construction and since ψ is an R_A–valid spacetime diagram, ψ' is an $R_{A'}$–valid spacetime diagram due to $R_\bullet$ gauge-invariance. Hence T and T' are equivalent.

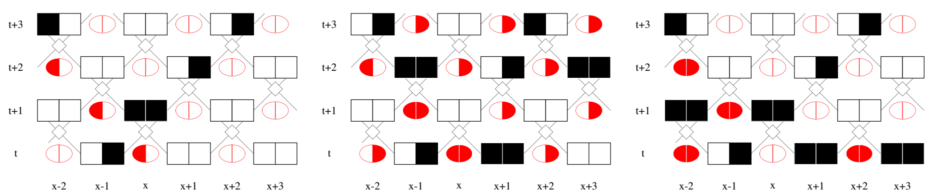
From here, we have a lot of possible dynamics $S_\bullet$ to describe the same physics and choosing between those is based upon convenience. However there exists canonical ways to do so. One way is called the Lorenz-gauge (well known in electromagnetism) which consists of taking the sum of the partial derivatives of the gauge-field as null (i.e. $\sum_i \partial_i A_i = 0$). The Lorenz-gauge is central in physics because it is the simplest gauge fixing rule to be invariant under boost and global rotation in spacetime, namely Lorentz invariant.

This constrains the dynamics because given an initial configuration A, we have $A_r(x, t+1) - A_r(x-1, t) = A_l(x, t+1) - A_l(x+1, t)$. An example of a Lorenz-gauge is given in Fig-4a. Therefore, when using the Lorenz-gauge, we fix the dynamics of the gauge field A while keeping a complete freedom on its initial condition.

To grasp the extent of the previous results, we will look at some examples. In all of the following examples we take the same initial condition for $c = (\psi, A)$ and the same dynamics R_A for ψ and we look at different dynamics S_ψ on A. First if we have different F (e.g. $F = 0$ and $F = 1$), then the resulting diagrams cannot be physically equivalent — e.g. Fig-4 (a) and (b) — even if we have the same $\widetilde{S}_\bullet$. Then, if we have the same F, even with different $S_\bullet$, we will have the same physical solutions (Fig-4 (a) and (c)) which is what was proven above. Finally, if we do not even take the same $\widetilde{S}_\bullet$ then we cannot have the same F and thus, the physical solutions are different (Fig-4 (a) and (d)).

Gauge-constraining : removing redundancies. Now that we fully described our theory, we still have some redundancies because of the initial configuration and possibly an incomplete gauge-fixing. For instance, two diagrams may use the same Lorenz-gauge (same dynamics) with two different initial configurations and we will still have the same physical solutions such as shown in Fig-(5a and 5b). Therefore we may want to restrict the space of configurations by adding some constraints directly on the fields, that is what we name gauge-constraining. This gauge-constraining may be done apart from gauge-fixing. However we must keep in mind that constraining the gauge field can also restrict the set of physical solutions available — e.g. if you choose $A_r = A_l = 0$ then there is no degree of freedom on A anymore and the set of physical solutions is limited.

One common way to gauge-constrain a gauge-field while conserving enough degree of freedom to keep the complete set of physical solutions is called Weyl-gauge (or temporal gauge) which means adding the constraint $A_0 = cst$ where $A_0 = (A_r - A_l)/2$ for us. Writing the constraint with our notations directly lead to $A_r = A_l + cst$ which leads to a constrained set of gauge-field $\{A \,|\, A_l = A_r\}$. An example is given Fig-5c with $A_r = A_l$ ($cst = 0$). While constraining the gauge field, we could think we loose the gauge-invariance : we restrict the set of possible solutions for the gauge-field, thus not every local transformation could be allowed inside this set. However, every item in the constrained set of solutions is also an element of the most general set, hence conserving the gauge-invariance. Therefore, when gauge-constraining, we fix a condition on the gauge-field directly while keeping a small degree of freedom on its initial condition and sometimes on its dynamics too — in the Weyl-gauge at each time-step there can be two choices : either black-black or white-white. An example is given Fig-5 where all three diagrams are equivalent with the last one having the Weyl-gauge as a constraint.



(a) Lorenz-gauge. (b) Another Lorenz-gauge. (c) Temporal gauge.

Fig. 5: Three physically equivalent spacetime diagrams with different initial conditions and gauges.

5 Conclusion

Summary. The paper followed a discrete counterpart to the gauging procedure, which aims to enforce a local symmetry that was judged missing in some physical theory. Here, theories were captured as Cellular Automata (CA), and local symmetries as local transformations G_φ of the spacetime diagrams c of these CA, i.e. mapping each cell $c(x,t)$ into $G_{\varphi(x,t)}c(x,t)$. Gauge-invariance was formulated as a concrete condition (1) upon the local rule of the theory. It was shown how, starting from a homogeneous non-gauge-invariant local rule R over configurations $\psi(.,t)$, one gets to an $A(x,t)$–dependent gauge-invariant local rule $R_\bullet$, and completes this with a $\psi(x,t)$–dependent gauge-invariant local rule $S_\bullet$ over configurations $A(.,t)$, in order to finally obtain a homogeneous gauge-invariant local rule $T = R_\bullet \cup S_\bullet$ over configurations $c(.,t) = (\psi(.,t), A(.,t))$. The acquired gauge-symmetry then leads to equivalent theories T'—equivalent up to the symmetry. A way to go from a T to some equivalent T' is to replace $S_\bullet$ by some $S'_\bullet$ whose spacetime diagrams are G_φ–related—this is called gauge-fixing. Theory equivalence and gauge-fixing were formalized, the fact that the latter respects the former was proven. Moreover, one can sometimes find an equivalent theory on a reduced configuration space $\widetilde{c}(.,t)$, which can be understood as a canonical representant of $c(.,t)$ under the symmetry—this is called gauge-constraining. We provided a simple, concrete instance of this, as well as all of the previous notions. The whole discrete gauge-invariance theory has been proved to be the discrete analogous of the modern classical electrodynamics.

*Motivations.*These were twofold: (i) Porting the gauge theoretical tools and concepts to Computer Science, as methods for constructing nature-inspired CA; providing more accurate schemes for numerical analysis; providing quantum simulation algorithms; making spatially distributed (quantum) computation immune to local errors. (ii) Clarifying the gauge theoretical concepts through the simplicity and rigor brought by Discrete Mathematics; providing the most direct route to its core, i.e. without reference to quantum mechanics and least action principle.

Related works. A number of discrete counterparts to physics symmetries have been reformulated in terms of CA, including reversibility, Lorentz-covariance[10], conservations laws and invariants[11], but no gauge symmetry. To our knowledge the closest work is the colour-blind CA construction[12] which implements a global colour symmetry without porting it to the local scale. However this has been done in the one-particle sector of Quantum CA, a.k.a for Quantum Walks. Indeed, one of the authors had followed a similar procedure in order to introduce the electromagnetic gauge field [13], and that of the weak and strong interactions [14,15]. This again was done in the very fabric of the Quantum Walk and the associated symmetry was therefore an intrinsic property of the Quantum Walk. But the gauge field would remain continuous, and seen as an external field.

There are, of course, numerous other approaches to space-discretized gauge theories, the main ones being Lattice Gauge Theory[16] and the Quantum Link Model[17], which were phrased in terms of Quantum Computation through Tensor Networks[18] and can be linked in a unified framework[19]. A discretized gauge-theory can also arise from Ising models[19,20]. All of these approaches begin with a well-known continuous gauge theory which is then space-discretized—time is usually kept continuous. An interesting attempt to quantum discretize gauge theories in discrete time on a general simplicial complex can be found in [21].

Perspectives. We are confident that the hereby developed methodology is ready to be applied to Quantum CA (QCA) [22], so as to obtain discretized free and interacting Quantum Field Theories [23]. Such discretized theories are of interest in Physics especially in non-perturbative theories [24], but they also represent practical assets as quantum simulation algorithms, i.e. numerical schemes that run on Quantum Computers to efficiently simulate interacting fundamental particles—a task which is beyond the capabilities of classical computers. This is ongoing work.

6 Acknowledgements

The authors would like to thank Cédric Bény, Thomas Krajewski, Terry Farrelly and Pablo Arnault, for very instructive conversations about gauge theories. This work was partially supported by the *CNRS PEPS JCJC*, ”GQNet” and *Dèfi InFinitTI 2018*, ”Lattice Quantum Simulation Theory” (LaQuST).

References

1. Quigg, C.: Gauge theories of the strong, weak, and electromagnetic interactions. Princeton University Press (2013)
2. Georgescu, I., Ashhab, S., Nori, F.: Quantum simulation. Reviews of Modern Physics **86**(1) (2014) 153
3. Hastings, W.K.: Monte carlo sampling methods using markov chains and their applications. Biometrika **57**(1) (1970) 97–109
4. Harao, M., Noguchi, S.: Fault tolerant cellular automata. Journal of computer and system sciences **11**(2) (1975) 171–185
5. Toom, A.: Cellular automata with errors: Problems for students of probability. Topics in Contemporary Probability and Its Applications (1995) 117–157
6. Kitaev, A.Y.: Fault-tolerant quantum computation by anyons. Annals of Physics **303**(1) (2003) 2–30
7. Nayak, C., Simon, S.H., Stern, A., Freedman, M., Sarma, S.D.: Non-abelian anyons and topological quantum computation. Reviews of Modern Physics **80**(3) (2008) 1083
8. Toffoli, T., Margolus, N.: Cellular Automata Machine – A new Environment for Modelling. MIT Press, Cambridge MA (1987)
9. Wolf-Gladrow, D.A.: Lattice-gas cellular automata and lattice Boltzmann models: an introduction. Springer (2004)
10. Arrighi, P., Facchini, S., Forets, M.: Discrete lorentz covariance for quantum walks and quantum cellular automata. New Journal of Physics **16**(9) (2014) 093007
11. Formenti, E., Kari, J., Taati, S.: On the hierarchy of conservation laws in a cellular automaton. Natural Computing **10**(4) (2011) 1275–1294
12. Salo, V., Trm, I.: Color blind cellular automata. Lecture Notes in Computer Science (2013) 139154
13. Di Molfetta, G., Brachet, M., Debbasch, F.: Quantum walks in artificial electric and gravitational fields. Physica A: Statistical Mechanics and its Applications **397** (2014) 157–168
14. Arnault, P., Di Molfetta, G., Brachet, M., Debbasch, F.: Quantum walks and non-abelian discrete gauge theory. Physical Review A **94**(1) (2016) 012335
15. Di Molfetta, G., Pérez, A.: Quantum walks as simulators of neutrino oscillations in a vacuum and matter. New Journal of Physics **18**(10) (2016) 103038
16. Willson, S.J.: Computing fractal dimensions for additive cellular automata. Physica D **24** (1987) 190–206
17. Chandrasekharan, S., Wiese, U.J.: Quantum link models: A discrete approach to gauge theories. Nuclear Physics B **492**(1-2) (1997) 455–471
18. Rico, E., Pichler, T., Dalmonte, M., Zoller, P., Montangero, S.: Tensor networks for lattice gauge theories and atomic quantum simulation. Physical Review Letters **112**(20) (2014) 201601
19. Silvi, P., Rico, E., Calarco, T., Montangero, S.: Lattice gauge tensor networks. New Journal of Physics **16**(10) (Oct 2014) 103015
20. Wegner, F.J.: Duality in generalized ising models and phase transitions without local order parameters. Journal of Mathematical Physics **12**(10) (Oct 1971) 22592272
21. Kornyak, V.V.: Discrete dynamics: gauge invariance and quantization. In: International Workshop on Computer Algebra in Scientific Computing, Springer (2009) 180–194
22. Arrighi, P., Nesme, V., Werner, R.: One-dimensional quantum cellular automata over finite, unbounded configurations. In: International Conference on Language and Automata Theory and Applications, Springer (2008) 64–75
23. Itzykson, C., Zuber, J.B.: Quantum field theory. Courier Corporation (2006)
24. Strocchi, F.: An introduction to non-perturbative foundations of quantum field theory. Volume 158. Oxford University Press (2013)