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On detectability of labeled Petri nets with inhibitor arcs

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Abstract Detectability is a basic property of dynamic systems: when it holds one can use the observed output signal produced by a system to reconstruct its current state.

In this paper, we consider properties of this type in the framework of discrete event systems modeled by Petri nets (a.k.a. place/transition nets). We first study weak detectability and weak approximate detectability. The former implies that there exists an evolution of the net such that all corresponding observed output sequences longer than a given value allow one to reconstruct the current marking (i.e., state). The latter implies that there exists an evolution of the net such that all corresponding observed output sequences longer than a given value allow one to determine if the current marking belongs to a given set. We show that the problem of verifying the first property for labeled place/transition nets with inhibitor arcs and the problem of verifying the second property for labeled place/transition nets are both undecidable.

We also consider a property called instant strong detectability which implies that for all possible evolutions the corresponding observed output sequence allows one to reconstruct the current marking. We show that the problem of verifying this property for labeled place/transition nets is decidable while its inverse problem is EXPSPACE-hard.

Keywords Labeled Petri net · Inhibitor arc · Weak detectability · Weak approximate detectability · Instant strong detectability · Decidability

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1 Introduction

Detectability is a basic property of dynamic systems: when it holds one can use the observed output signal produced by a system to reconstruct its current state [6,25,23, 24,4,27,34,17,30,12]. This property plays a fundamental role in many related control problems such as observer design and controller synthesis. Hence for different applications, it is meaningful to characterize detectability for control systems in different frameworks. This property also has different terminology, e.g., in [6,27,17], it is called "observability"; in [4,34], it is called "reconstructibility". In this paper, we uniformly call it "detectability", and call the property whether the initial state can be determined by the observed output signal produced by a system "observability".

For *discrete event systems* (DESs), the detectability problem in the framework of *finite automata* has been widely studied [25,23,24,31,12,30]. For different uses, detectability is formulated as *strong detectability* and *weak detectability* [25], where the former means in the ω*-language* (i.e., the set of infinite observed output sequences) generated by a DES, whether there exists a positive integer k such that each prefix of each infinite output sequence of the ω-language longer than k can be used to reconstruct the current state, while the latter means in the ω-language, whether there exists an infinite output sequence such that each of its prefixes longer than a positive integer l can be used to do that. Strong detectability can be verified in polynomial time but weak detectability can only be verified in exponential time currently [25,23]. Checking weak detectability of DESs is PSPACE-complete in the numbers of states and events even for deterministic DESs whose events are all observable [31], hence it is unlikely that there exists a polynomial time algorithm for verifying weak detectability. Other related results on the complexity of deciding detectability of DESs can be found in [30,12].

What if the framework of *labeled Petri nets* (a.k.a. *labeled place/transition nets* (labeled P/T nets)) is considered? Although labeled P/T nets have finitely many transitions (i.e., events), they may have countably infinitely many markings (i.e., states). Hence the detectability for labeled P/T nets may be more complex. Taking opacity for example, where opacity is a property whether an intruder (outside a system) can never determine whether some states of the system prior to the current time step are secret, although verifying different types of opacity of finite-automaton-based DESs are at least NP-hard in the numbers of states and events, they are decidable [20,19,18,21] (stochastic finite automata excluded [22]). However, checking opacity is generally undecidable [1,8,26]. Then it is interesting to study whether from the perspective of detectability, whether Petri-net-based DESs are more complex than finite-automaton-based DESs. In this paper, we obtain several results.

As stated before, weak detectability roughly means reconstructing the current marking by using observed *labeling* (i.e., output) sequences. Sometimes, we do not need to reconstruct the current state but only need to estimate whether the current state belongs to a given subset of all reachable markings. For example, we determine whether the current state is normal or not (here we do not need to give a physical description for "normality" and "abnormality", we just need to know they stand for different types of reachable markings.). In this sense, the set of all reachable markings is partitioned into two disjoint subsets: the subset of normal reachable markings and the set of abnormal reachable markings. We call such a detectability *weak approximate detectability*. In this paper, we will prove that the weak approximate detectability of labeled P/T nets is undecidable[1] for any fixed finite cardinality of partition of reachable markings. On the other hand, when *inhibitor arcs* are added into labeled P/T nets, we prove that the weak detectability of labeled P/T nets with inhibitor arcs is undecidable. These undecidable results are obtained by reducing the well known undecidable *language equivalence problem* [7] for labeled P/T nets to the problems under consideration.

As we have already mentioned, strong detectability implies that there exists a finite integer k such that each prefix of length greater than k of each infinite labeling sequence of the ω-language generated by a labeled Petri net allows one to reconstruct the current marking. However, when we do synthesis for a labeled P/T net, we wish that the marking can be determined once the net started to run. In this sense, we study a new detectability that is stronger than the previous strong detectability which we call *instant strong detectability*, which means that whether each labeling sequence generated by the net can determine the current marking. Actually, a stronger version of instant strong detectability has been studied in [17], where it is called "structural observability", since it implies that the instant strong detectability is satisfied for all initial markings. It is pointed out that the "structural observability" is important, because "the majority of existing control schemes for Petri nets rely on complete knowledge of the system state at any given time step" [17]. In [17], the optimal problem of placing the minimal number of sensors on places/transitions to make a labeled Petri net structurally observable is studied. The former problem is proved to be NP-complete, while the latter is shown to be solvable in polynomial time, both in the numbers of places and transitions. However, the decidability of instant strong detectability has not been studied yet. In this paper, we will prove that the instant strong detectability problem is decidable by reducing it to the known decidable *home space problem* [2] of Petri nets with respect to a computable *semi-linear subset* [14] and deciding its inverse problem is EXPSPACE-hard in the numbers of places and transitions of the labeled P/T net and the number of tokens of the destination marking in the coverability problem. by showing that the EXPSPACE-complete *coverability problem* [15,11] is polynomial time reducible to the non-instant strong detectability problem. This home space problem has been used to prove several decidable results on several other types of detectability of P/T nets with unknown initial markings [6], where these types of detectability are called (strong) marking observability, uniform (strong) marking observability, and structural (strong) marking observability. Note that these types of detectability are decidable may be partially due to that the Petri nets considered in [6] are unlabeled, although their initial markings are unknown.

The contributions of this paper are as follows: We prove that 1) the weak detectability of ϵ-free (i.e., no transition is labeled as the empty word ϵ) labeled P/T nets with inhibitor arcs is undecidable, 2) the weak approximate detectability of ϵ-free labeled P/T nets is undecidable even restricted to a fixed finite cardinality of partition of the set of reachable markings, and 3) the instant strong detectability of

[1] This sentence is short for "The problem of verifying the weak approximate detectability of labeled P/T nets is undecidable". In the sequel, we will always use such a short expression. That is, we use "a property is undecidable" instead of "the problem of verifying the property is undecidable."

labeled P/T nets is decidable and its inverse problem is EXPSPACE-hard in the numbers of places and transitions of the labeled P/T net and the number of tokens of the destination marking in the coverability problem.

The remainder of the paper is arranged as follows. Section 2 introduces necessary preliminaries, Section 3 shows the main results, and Section 4 ends up with a short conclusion.

2 Preliminaries

For a finite set S, S^* and S^ω are used to denote the sets of finite sequences (called *words*) of elements of S including the empty word ϵ and infinite sequences (called *configurations*) of elements of S, respectively. For a word $s \in S^*$, $|s|$ stands for its length, and we set $|s'| = +\infty$ for all $s' \in S^\omega$. For $s \in S$ and natural number k, s^k and s^ω denote the k-length word and configuration consisting of copies of s's, respectively. For a word (configuration) $s \in S^*(S^\omega)$, a word $s' \in S^*$ is called a *prefix* of s, denoted as $s' \sqsubseteq s$, if there exists another word (configuration) $s'' \in S^*(S^\omega)$ such that $s = s's''$. For two natural numbers $i \leq j$, $[i, j]$ denotes the set of all integers between i and j; and for a set S, $|S|$ its cardinality.

A *net* is a quadruple $N = (P, T, Pre, Post)$, where P is a finite set of *places* graphically represented by circles; T is a finite set of *transitions* graphically represented by bars; $P \cup T \neq \emptyset$, $P \cap T = \emptyset$; $Pre : P \times T \to \mathbb{N}$ and $Post : P \times T \to \mathbb{N}$ are the *pre-* and *post-incidence functions* that specify the arcs directed from places to transitions, and vice versa, where $\mathbb{N}$ stands for the set of natural numbers. The *incidence function* is defined as $C = Post - Pre$.

A *marking* is a map $M : P \to \mathbb{N}$ that assigns to each place of a net a natural number of tokens, graphically represented by black dots. For a marking $M \in \mathbb{N}^P$, a transition $t \in T$ is called *enabled* at M if $M(p) \geq Pre(p, t)$ for any $p \in P$, and is denoted by $M[t\rangle$, where as usual $\mathbb{N}^P$ denotes the set of maps from P to $\mathbb{N}$. An enabled transition t at M may *fire* and yield a new making $M'(p) = M(p) + C(p, t)$ for all $p \in P$, written as $M[t\rangle M'$. As usual, we assume that at each marking and each time step, at most one transition fires. For a marking M, a sequence $t_1 \dots t_n$ of transitions is called enabled at M if t_1 is enabled at M, t_2 is enabled at the unique M_2 satisfying $M[t_1\rangle M_2$, $\dots$, t_n is enabled at the unique M_{n-1} satisfying $M[t_1\rangle \cdots [t_{n-1}\rangle M_{n-1}$. We write the firing of $t_1 \dots t_n$ at M as $M[t_1 \dots t_n\rangle$ for short, and similarly denote the firing of $t_1 \dots t_n$ at M yielding M' by $M[t_1 \dots t_n\rangle M'$. $\mathcal{T}(N, M_0) := \{s \in T^* | M_0[s\rangle\}$ is used to denote the set of transition sequences enabled at M_0. Particularly we have $M_0[\epsilon\rangle M_0$. A pair (N, M_0) is called a *Petri net* or a *place/transition net (P/T net)*, where $N = (P, T, Pre, Post)$ is a net, $M_0 : P \to \mathbb{N}$ is called the *initial marking*, and the Petri net evolves initially at M_0 as transition sequences fire. Denote the set of *reachable markings* of the Petri net by $\mathcal{R}(N, M_0) := \{M \in \mathbb{N}^P | \exists s \in T^*, M_0[s\rangle M'\}$. For a Petri net (N, M_0), $\mathcal{R}(N, M_0)$ is at most countably infinite.

A *labeled P/T net* is a quadruple (N, M_0, Σ, ℓ), where N is a net, M_0 is an initial marking, Σ is an *alphabet* (a finite set of labels), and $\ell : T \to \Sigma \cup \{\epsilon\}$ is a *labeling function* that assigns to each transition $t \in T$ a symbol of Σ or the empty word ϵ, which means when a transition t fires, its label $\ell(t)$ can be observed if $\ell(t) \in \Sigma$; and nothing can be observed if $\ell(t) = \epsilon$. Particularly, a labeling function $\ell : T \to \Sigma$ is called *ϵ-free*, and a P/T net with an ϵ-free labeling function is called an *ϵ-free labeled P/T net*. A Petri net is actually an ϵ-free labeled P/T net with an injective labeling function. A labeling function $\ell : T \to \Sigma \cup \{\epsilon\}$ can be recursively extended to $\ell : T^* \to \Sigma^*$ as $\ell(st) = \ell(s)\ell(t)$ for all $s \in T^*$ and $t \in T$. Particularly we let $\ell(\epsilon) = \epsilon$. For a labeled P/T net $G = (N, M_0, \Sigma, \ell)$, the *language generated by* G is denoted by $\mathcal{L}(G) := \{\sigma \in \Sigma^* | \exists s \in T^*, M_0[s\rangle, \ell(s) = \sigma\}$, i.e., the set of labels of finite transition sequences enabled at the initial marking M_0. We also say for each $\sigma \in \mathcal{L}(G)$, G generates σ. For $\sigma \in \Sigma^\omega$, we say G generates σ if G generates each prefix of σ. The set of configurations generated by G (i.e., the *ω-language*) is denoted by $\mathcal{L}^\omega(G)$.

Note that for a labeled P/T net $G = (N, M_0, \Sigma, \ell)$, when we observe a label sequence $\sigma \in \Sigma^*$, there may exist infinitely many firing transition sequences labeled by σ. However, for an ϵ-free labeled P/T net, when we observe a label sequence σ, there exist at most finitely many firing transition sequences labeled by σ. Denote by $\mathcal{M}(G, \sigma) := \{M \in \mathbb{N}^P | \exists s \in T^*, M_0[s\rangle M, \ell(s) = \sigma\}$, the set of markings in which G can be when σ is observed. Then for each $\sigma \in \Sigma^*$, $\mathcal{M}(G, \sigma)$ is finite for an ϵ-free labeled P/T net G.

So far we have considered labeled nets whose underlying structures are P/T nets. We will also consider a larger class of labeled nets whose underlying structures are P/T nets with inhibitor arcs. Formally a *net with inhibitor arcs* is a quintuple $N' = (P, T, Pre, Pre', Post)$, where P and T are also finite sets of places and transitions such that $P \cup T \neq \emptyset$ and $P \cap T = \emptyset$, $Pre : P \times T \to \mathbb{N}$ and $Post : P \times T \to \mathbb{N}$ are still the pre- and post-incidence functions, $Pre' : P \times T \to \{0, 1\}$ is the *inhibitor pre-incidence function* such that $Pre(p, t) \cdot Pre'(p, t) = 0$ for all $p \in P$ and $t \in T$, guaranteeing that there exists at most one of a normal arc and an inhibitor arc from p to t. Here a transition $t \in T$ is enabled at a marking $M \in \mathbb{N}^P$ if and only if $M(p) \geq Pre(p, t)$ for any $p \in P$ satisfying $Pre(p, t) > 0$ and $M(p) = 0$ for any $p \in P$ satisfying $Pre'(p, t) > 0$. The firing of a transition $t \in T$ at a marking $M \in \mathbb{N}^P$ yields a marking $M'(p) = M(p) + Post(p, t) - Pre(p, t)$ if $Pre'(p, t) = 0$ and $M'(p) = Post(p, t)$ if $Pre'(p, t) > 0$, where $p \in P$ and $t \in T$. Similarly, a *labeled P/T net with inhibitor arcs* is a quadruple $G' = (N', M_0, \Sigma, \ell)$, where $N' = (P, T, Pre, Pre', Post)$ is a net with inhibitor arcs, $M_0 \in \mathbb{N}^P$ is an initial marking, Σ is again an alphabet, and $\ell : T \to \Sigma \cup \{\epsilon\}$ is again a labeling function. The set $\mathcal{T}(N', M_0)$ of finite transition sequences enabled at M_0, the set $\mathcal{R}(N', M_0)$ of reachable markings, the language $\mathcal{L}(G')$ generated by G', and the set $\mathcal{M}(G', \sigma)$ of markings in which G' can be when $\sigma \in \Sigma^*$ is observed, are defined in an analogue way as those for labeled P/T nets.

The undecidable results obtained in this paper is obtained by using the following language equivalence problem.

Proposition 1 *[7, Theorem 8.2] It is undecidable to verify whether two ϵ-free labeled P/T nets with the same alphabet generate the same language.*

On the other hand, the decidable results obtained in this paper are obtained by the home space problem with respect to a semi-linear subset [2]. Necessary preliminaries are introduced as follows.

Given a finite set P, a subset $E \subseteq \mathbb{N}^P$ is called *linear* [14] if there exist $v, v_1, \dots, v_n \in \mathbb{N}^P$ such that

$$E = \left\{ v + \sum_{i=1}^{n} k_i v_i | k_1, \dots, k_n \in \mathbb{N} \right\},$$

where v is usually called *base* and $v_1, \dots, v_n$ are called *periods*. More generally, a subset $F \subseteq \mathbb{N}^P$ is called *semi-linear* if it is a finite union of linear subsets $E_1, \dots, E_q$ of $\mathbb{N}^P$, and we call the bases and periods of $E_1, \dots, E_q$ the bases and periods of F. We will use the helpful proposition as below.

Proposition 2 *[5, Theorem 6.2] If X and Y are semi-linear subsets of $\mathbb{N}^P$, then $X \setminus Y$ is also a semi-linear subset of $\mathbb{N}^P$ and is effectively calculable from X and Y.*

By this proposition, given semi-linear subsets X and Y of $\mathbb{N}^P$, one can use the constructive proof (which is an algorithm) to compute the base and periods of $X \setminus Y$ as a semi-linear subset of $\mathbb{N}^P$ from the bases and periods of X and Y. The concept of semi-linear subsets is closely related to Petri nets [16,2,6].

The decidable result on the home space problem with respect to a semi-linear subset E is as shown in Proposition 3. Note that when E reduces to a singleton (e.g., when all periods of E are the zero vector and all bases of the finitely many linear subsets whose union equals E are the same), the home space problem becomes the well known *reachability problem*. The reachability problem is decidable [13,9,10], and EXPSPACE-hard [11]. The home space problem with respect to a linear subset can be reduced to the reachability problem with respect to the base of the linear subset [2]. Furthermore in [2], an algorithm for solving the home space problem with respect to a semi-linear subset is constructed by using the algorithm for solving the home space problem with respect to a linear subset.

Proposition 3 *[2, Corollary 1] It is decidable to verify for a Petri net (N, M_0) and a semi-linear subset E of $\mathbb{N}^P$ whether $\mathcal{R}(N, M_0) \cap E = \emptyset$.*

We also need the following Proposition 4 on the coverability problem as below to obtain another main result.

Proposition 4 *[15,11] It is EXPSPACE-complete to decide for a Petri net $G = (N, M_0)$ and a destination marking $M \in \mathbb{N}^P$ if G covers M, i.e., if there exists a marking $M' \in \mathcal{R}(N, M_0)$ such that $M \le M'$, i.e., $M(p) \le M'(p)$ for each place p of N.*

In [11], it is proved that deciding the coverability for Petri nets requires at least 2^{cn} space infinitely often for some constant $c > 0$, where n is the number of transitions. While in [15], it is shown that deciding the same property for a Petri net requires at most space $2^{cm \log m}$ for some constant c, where m is the size of the set of all transitions of the Petri net. For a Petri net $((P, T, Pre, Post), M_0)$, each transition $t \in T$ corresponds to a $|P|$-length vector $Post(\cdot, t) - Pre(\cdot, t) =: c(t)$ whose components are integers. The size of t is the sum of the lengths of the binary representations of the components of $c(t)$ (where the length of 0 is 1). The size of T is the sum of the sizes of all transitions of T, and is set to be the above m.

The coverability problem belongs to EXPSPACE [15]. However, it is not known whether the reachability problem belongs to EXPSPACE [3]. Using a similar reduction as the one in [2] used to reduce the home space problem with respect to a linear subset to the reachability problem with respect to the base of the linear subset, one can reduce the coverability problem to the reachability problem with respect to the same marking. Proposition 4 has been used to prove the EXPSPACE-hardness of checking non-diagnosability [29] and non-prognosability [28] of labeled Petri nets.

3 Main results

3.1 Weak detectability and weak approximate detectability

The concept of weak detectability is formulated as follows.

Definition 1 Consider a labeled P/T net G (with or without inhibitor arcs). The net G is called *weakly detectable* if there exists a label sequence $\sigma \in \Sigma^\omega$ such that for some positive integer k, $|\mathcal{M}(G, \sigma')| = 1$ for any prefix σ' of σ satisfying $|\sigma'| \geq k$.

Sometimes, we do not need to determine the current marking of a labeled net, but only need to know whether the current marking belongs to some prescribed subset of reachable markings. Then the concept of weak approximate detectability is formulated as below.

Definition 2 Consider a labeled P/T net G (with or without inhibitor arcs). Given a positive integer $n > 1$ and a partition $\{R_1, \dots, R_n\}$ of the set of its reachable markings, G is called *weakly approximately detectable* with respect to n and partition $\{R_1, \dots, R_n\}$ if there exists a label sequence $\sigma \in \Sigma^\omega$ such that for some positive integer k, for any prefix σ' of σ satisfying $|\sigma'| \geq k$, $\emptyset \neq \mathcal{M}(G', \sigma') \subseteq R_i$ for some $i \in [1, n]$ depending on σ'.

For weak detectability of labeled P/T nets with inhibitor arcs, we have the following results.

Theorem 1 *The weak detectability of ϵ-free labeled P/T nets with inhibitor arcs is undecidable.*

Proof We prove this result by reducing the language equivalence problem of ϵ-free labeled P/T nets (Proposition 1) to the weak detectability problem of ϵ-free labeled P/T nets with inhibitor arcs.

Arbitrarily given two ϵ-free labeled P/T nets $G_i = (N_i, M_0^i, \Sigma, \ell_i)$, where $N_i = (P_i, T_i, Pre_i, Post_i)$, $i = 1, 2$, $P_1 \cap P_2 = \emptyset$, $T_1 \cap T_2 = \emptyset$, we next construct a new ϵ-free labeled P/T net G with inhibitor arcs from G_1 and G_2, and prove that $\mathcal{L}(G_1) = \mathcal{L}(G_2)$ if and only if G is not weakly detectable.

In G, we add five more places and several new transitions to G_1 and G_2, where p_0 contains a unique token and (together with related transitions) starts G; p_1^1 (p_1^2) starts G_1 (G_2); after G_1 (G_2) stops, p_2^1 (p_2^2) is used to clear all places of G_1 (G_2). G is specified as $(N_G, M_0^G, \Sigma \cup \{\sigma_G\}, \ell_G)$ (see Fig. 1 as a sketch), where

1. $N_G = (P_G, T_G, Pre_G, Pre'_G, Post_G)$;
2. $P_G = P_1 \cup P_2 \cup \{p_0, p_1^1, p_1^2, p_2^1, p_2^2\}$, $(P_1 \cup P_2) \cap \{p_0, p_1^1, p_1^2, p_2^1, p_2^2\} = \emptyset$, $|\{p_0, p_1^1, p_1^2, p_2^1, p_2^2\}| = 5$;
3. $T_G = T_1 \cup T_2 \cup \{t_0^1, t_0^2, t_1^1, t_1^2, t_1, t_2\} \cup \{t_p | p \in P_1\} \cup \{t_p | p \in P_2\}$, $(T_1 \cup T_2) \cap (\{t_0^1, t_0^2, t_1^1, t_1^2, t_1, t_2\} \cup \{t_p | p \in P_1\} \cup \{t_p | p \in P_2\}) = \emptyset$, $|\{t_0^1, t_0^2, t_1^1, t_1^2, t_1, t_2\} \cup \{t_p | p \in P_1\} \cup \{t_p | p \in P_2\}| = 6 + |P_1| + |P_2|$;
4. $Pre_G|_{P_1 \times T_1}$(the restriction of Pre_G to $P_1 \times T_1$) $= Pre_1$, $Pre_G|_{P_2 \times T_2} = Pre_2$, $Pre_G(p_0, t_0^1) = Pre_G(p_0, t_0^2) = Pre_G(p_1^1, t_1^1) = Pre_G(p_1^2, t_1^2) = 1$, $Pre_G(p_1^1, t)$ $= 1$ for any $t \in T_1$, $Pre_G(p_1^2, t) = 1$ for any $t \in T_2$, $Pre_G(p_2^1, t_p) = 1$ for any $p \in P_1$, $Pre_G(p_2^2, t_p) = 1$ for any $p \in P_2$, $Pre_G(p_2^1, t_1) = Pre_G(p_2^2, t_2) = 1$, $Pre_G(p, t_p) = 1$ for any $p \in P_1 \cup P_2$, $Pre_G(p, t) = 0$ for any other $(p, t) \in P_G \times T_G$;
5. $Pre'_G(p, t_1) = 1$ for any $p \in P_1$, $Pre'_G(p, t_2) = 1$ for any $p \in P_2$, $Pre'_G(p, t) =$ 0 for any other $(p, t) \in P_G \times T_G$;
6. $Post_G|_{P_1 \times T_1} = Post_1$, $Post_G|_{P_2 \times T_2} = Post_2$, $Post_G(p_1^1, t_0^1) = Post_G(p_1^2, t_0^2)$ $= Post_G(p_2^1, t_1^1) = Post_G(p_2^2, t_1^2) = 1$, $Post_G(p_1^1, t) = 1$ for any $t \in T_1$, $Post_G(p_1^2, t) = 1$ for any $t \in T_2$, $Post_G(p_2^1, t_p) = 1$ for any $p \in P_1$, $Post_G(p_2^2, t_p)$ $= 1$ for any $p \in P_2$, $Post_G(p_2^1, t_1) = Post_G(p_2^2, t_2) = 1$, $Post_G(p, t) = 0$ for any other $(p, t) \in P_G \times T_G$;
7. $M_0^G \in \mathbb{N}^{P_G}$ is specified as $M_0^G|_{P_i} = M_0^i$, $i = 1, 2$, $M_0^G(p_0) = 1$, $M_0^G(p_1^1) = M_0^G(p_1^2) = M_0^G(p_2^1) = M_0^G(p_2^2) = 0$;
8. σ_G is a new symbol, i.e., $\sigma_G \notin \Sigma$;
9. $\ell_G|_{T_1} = \ell_1$, $\ell_G|_{T_2} = \ell_2$, $\ell_G(t_0^1) = \ell_G(t_0^2) = \ell_G(t_1^1) = \ell_G(t_1^2) = \ell_G(t_1) = \ell_G(t_2) = \sigma_G$, $\ell_G(t_p) = \sigma_G$ for any $p \in P_1 \cup P_2$.

For net G, initially only transition t_0^1 or t_0^2 can fire. After t_0^1 (t_0^2) fires, the unique token in place p_0 moves to place p_1^1 (p_1^2), initializing net G_1 (G_2). While G_1 (G_2) is running, only transition t_1^1 (t_1^2) outside $T_1 \cup T_2$ can fire. The firing of t_1^1 (t_1^2) moves the token in place p_1^1 (p_1^2) to place p_2^1 (p_2^2), and stops G_1 (G_2) from running, yielding that G_1 (G_2) will never run again, and for each $p \in P_1$ ($p \in P_2$), transition t_p fires repetitively until there exists no token in place p. After all places in P_1 (P_2) become empty, only transition t_1 (t_2) can fire, and can fire repetitively forever. All in all, all possible infinite transition sequences fired by G are of the form $t_0^1 s t_1^1 s' t_1^\omega$, $t_0^1 s''$, $t_0^2 r t_1^2 r' t_2^\omega$, or $t_0^2 r''$, where $s \in (T_1)^*$, $s' \in \{t_p | p \in P_1\}^*$, $s'' \in (T_1)^\omega$, $r \in (T_2)^*$, $r' \in \{t_p | p \in P_2\}^*$, $r'' \in (T_2)^\omega$. Note that for some G_1 and G_2, the corresponding G never fires $t_0^1 s''$ or $t_0^2 r''$ as above, e.g., when $\mathcal{L}(G_1) \cup \mathcal{L}(G_2)$ is finite; but for all G_1 and G_2, the corresponding G fires $t_0^1 s t_1^1 s' t_1^\omega$ and $t_0^2 r t_1^2 r' t_2^\omega$ as above.

If $\mathcal{L}(G_1) \neq \mathcal{L}(G_2)$, without loss of generality, we assume that there exists $\sigma \in \mathcal{L}(G_1) \setminus \mathcal{L}(G_2)$. Then when G generates $\sigma_G \sigma (\sigma_G)^\omega$, it only fires $t_0^1 s t_1^1 s' (t_1)^\omega$, where $s \in (T_1)^*$, $\ell_G(s) = \sigma$, $s' \in \{t_p | p \in P_1\}^*$, $|s'| = \sum_{p \in P_1} M(p)$, $M \in \mathbb{N}^{P_1}$ is

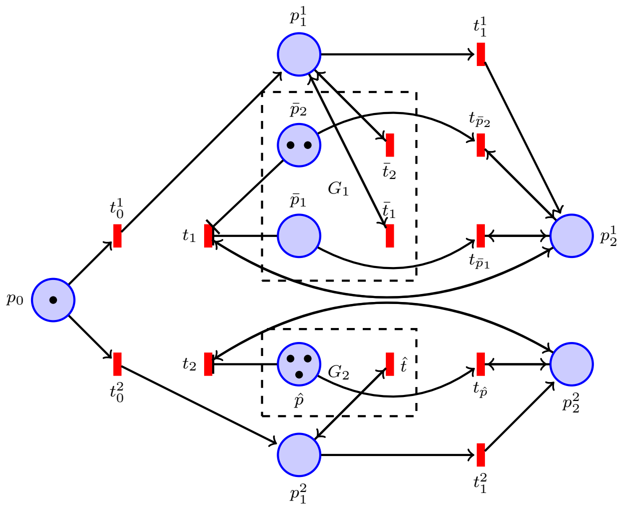

Fig. 1 Sketch for the reduction in the proof of Theorem 1, where all transitions outside $G_1 \cup G_2$ are with the same label.

the marking satisfying $M_0^1[s\rangle M$ uniquely determined by s. When we observe prefix $\sigma_G\sigma(\sigma_G)^k$ of $\sigma_G\sigma(\sigma_G)^\omega$ for any integer $k > K := \max\{\sum_{p\in P_1} M'(p)|\exists\tilde{s} \in (T_1)^*, \ell_G(\tilde{s}) = \sigma, M_0^1[\tilde{s}\rangle M'\}$ (note that $\{\tilde{s} \in (T_1)^*|\ell_G(\tilde{s}) = \sigma, M_0^1[\tilde{s}\rangle\}$ is a finite set, hence K is a natural number), the set $\mathcal{M}(G, \sigma_G\sigma(\sigma_G)^k)$ of reachable markings of G after observing $\sigma_G\sigma(\sigma_G)^k$ is a singleton, and its unique element $M_G \in \mathbb{N}^{P_G}$ satisfies that $M_G(p_0) = M_G(p_1^1) = M_G(p_1^2) = M_G(p_2^2) = M_G(p) = 0$ for any $p \in P_1$, $M_G(p_2^1) = 1$, $M_G|_{P_2} = M_0^2$. Hence G is weakly detectable.

If $\mathcal{L}(G_1) = \mathcal{L}(G_2)$, then G may generate only configurations $\sigma_G\sigma'$ or $\sigma_G\sigma(\sigma_G)^\omega$, where $\sigma' \in \Sigma^\omega$, $\sigma \in \mathcal{L}(G_1)$. For the former case, for any positive integer k and any k length prefix σ'' of σ', there exist firing sequences $s \in (T_1)^*$ of G_1 and $r \in (T_2)^*$ of G_2 such that $\ell_G(s) = \ell_G(r) = \sigma''$. Then $\mathcal{M}(G, \sigma'')$ includes a marking $M_G \in \mathbb{N}^{P_G}$ satisfying $M_G(p_1^1) = 1$ and $M_G(p_1^2) = 0$ and also a marking $M'_G \in \mathbb{N}^{P_G}$ satisfying $M'_G(p_1^1) = 0$ and $M'_G(p_1^2) = 1$. That is, $\mathcal{M}(G, \sigma'')$ is not a singleton. For the latter case, when we observe $\sigma_G\sigma(\sigma_G)^k$, where k is a sufficiently large natural number, we have G may fire both $t_0^1 s t_1^1 s'(t_1)^{k-1-|s'|}$ and $t_0^2 r t_1^2 r'(t_2)^{k-1-|r'|}$, where $s \in (T_1)^*$, $r \in (T_2)^*$, $\ell_G(s) = \ell_G(r) = \sigma$, $s' \in \{t_p|p \in P_1\}^*$, $r' \in \{t_p|p \in P_2\}^*$, $|s'| \le k-1$, $|r'| \le k-1$. Then we obtain two markings $M_G, M'_G \in \mathbb{N}^{P_G}$ satisfying that $M_0^G[t_0^1 s t_1^1 s'(t_1)^{k-1-|s'|}\rangle M_G$ and $M_0^G[t_0^2 r t_1^2 r'(t_2)^{k-1-|r'|}\rangle M'_G$, $M_G(p_2^1) = 1$, $M_G(p_2^2) = 0$, $M'_G(p_2^1) = 0$, $M'_G(p_2^2) = 1$. That is, $\mathcal{M}(G, \sigma_G\sigma(\sigma_G)^k)$ is not a singleton for any sufficiently large k. We have checked all label sequences generated by G, hence G is not weakly detectable, which completes the proof.

For the weakly approximate detectability of labeled P/T nets, the following result holds.

Theorem 2 *Let $n > 1$ be a positive integer. It is undecidable to verify for an ϵ-free labeled P/T net and a partition $\{R_1, \dots, R_n\}$ of the set of its reachable markings, whether the labeled P/T net is weakly approximately detectable with respect to n and $\{R_1, \dots, R_n\}$.*

Proof We prove this result also by reducing the language equivalence problem of labeled Petri nets (Proposition 1) to the problem under consideration. The proof is divided into three cases: $n = 2$, $n = 3$, and $n > 3$. The first two cases are based on the same reduction, the third part is based on a more complex reduction.

$n = 2$:

Arbitrarily given two ϵ-free labeled P/T nets $G_i = (N_i, M_0^i, \Sigma, \ell_i)$, where $N_i = (P_i, T_i, Pre_i, Post_i)$, $i = 1, 2$, $P_1 \cap P_2 = \emptyset$, $T_1 \cap T_2 = \emptyset$, we next construct an ϵ-free labeled P/T net G from G_1 and G_2 and a partition $\{R_1, R_2\}$ of the set of its reachable markings, and prove that $\mathcal{L}(G_1) = \mathcal{L}(G_2)$ if and only if G is not weakly approximately detectable with respect to 2 and partition $\{R_1, R_2\}$.

G is specified as $(N_G, M_0^G, \Sigma \cup \{\sigma_G\}, \ell_G)$ (see Fig. 2 as a sketch), where

1. $N_G = (P_G, T_G, Pre_G, Post_G)$;
2. $P_G = P_1 \cup P_2 \cup \{p_0, p_1^1, p_1^2\}$, $(P_1 \cup P_2) \cap \{p_0, p_1^1, p_1^2\} = \emptyset$, $|\{p_0, p_1^1, p_1^2\}| = 3$;
3. $T_G = T_1 \cup T_2 \cup \{t_0^1, t_0^2, t_1^1, t_1^2\}$, $(T_1 \cup T_2) \cap \{t_0^1, t_0^2, t_1^1, t_1^2\} = \emptyset$, $|\{t_0^1, t_0^2, t_1^1, t_1^2\}| = 4$;
4. $Pre_G|_{P_1 \times T_1} = Pre_1$, $Pre_G|_{P_2 \times T_2} = Pre_2$, $Pre_G(p_0, t_0^1) = Pre_G(p_0, t_0^2) = Pre_G(p_1^1, t_1^1) = Pre_G(p_1^2, t_1^2) = 1$, $Pre_G(p_1^1, t) = 1$ for any $t \in T_1$, $Pre_G(p_1^2, t) = 1$ for any $t \in T_2$, $Pre_G(p, t) = 0$ for any other $(p, t) \in P_G \times T_G$;
5. $Post_G|_{P_1 \times T_1} = Post_1$, $Post_G|_{P_2 \times T_2} = Post_2$, $Post_G(p_1^1, t_0^1) = Post_G(p_1^1, t_1^1) = 1$, $Post_G(p_1^2, t_0^2) = Post_G(p_1^2, t_1^2) = 1$, $Post_G(p_1^1, t) = 1$ for any $t \in T_1$, $Post_G(p_1^2, t) = 1$ for any $t \in T_2$, $Post_G(p, t) = 0$ for any other $(p, t) \in P_G \times T_G$;
6. $M_0^G \in \mathbb{N}^{P_G}$ is specified as $M_0^G|_{P_i} = M_0^i$, $i = 1, 2$, $M_0^G(p_0) = 1$, $M_0^G(p_1^1) = M_0^G(p_1^2) = 0$;
7. σ_G is a new symbol, i.e., $\sigma_G \notin \Sigma$;
8. $\ell_G|_{T_1} = \ell_1$, $\ell_G|_{T_2} = \ell_2$, $\ell_G(t_0^1) = \ell_G(t_0^2) = \ell_G(t_1^1) = \ell_G(t_1^2) = \sigma_G$.

For net G, initially only transition t_0^1 or t_0^2 can fire. After t_0^1 (t_0^2) fires, the unique token in place p_0 moves to place p_1^1 (p_1^2), initializing G_1 (G_2). While G_1 (G_2) is running, only transition t_1^1 (t_1^2) outside $T_1 \cup T_2$ can fire, and can fire infinitely many times. Transition $t_1^1(t_1^2)$ and transitions in $T_1(T_2)$ can fire alternatively. It can be seen that G can fire only infinite transition sequences $t_0^1 s_1 s_1' \dots s_k s_k' s''$ or $t_0^2 r_1 r_1' \dots r_k r_k' r''$, where $k \in \mathbb{N}$; $s_i \in (T_1)^*$, $s_i' \in \{t_1^1\}^*$, $r_i \in (T_2)^*$, $r_i' \in \{t_1^2\}^*$, $i \in [1, k]$; $s'' \in (T_1)^\omega \cup \{t_1^1\}^\omega$; $r'' \in (T_2)^\omega \cup \{t_1^2\}^\omega$. Hence the only possible configurations generated by G are of the form

$$\sigma_G \sigma_1 \sigma_1' \dots \sigma_k \sigma_k' \sigma, \tag{1}$$

where $k \in \mathbb{N}$; $\sigma_i \in \Sigma^*$, $\sigma_i' \in \{\sigma_G\}^*$, $i \in [1, k]$; $\sigma \in \Sigma^\omega \cup \{\sigma_G\}^\omega$; $\sigma_1 \dots \sigma_k \in \mathcal{L}(G_1) \cup \mathcal{L}(G_2)$; and $\sigma_1 \dots \sigma_k \sigma$ is generated by G_1 or G_2 if $\sigma \in \Sigma^\omega$. Note also that for some G_1 and G_2 and the corresponding G, the above σ cannot belong to Σ^ω, e.g., when $\mathcal{L}(G_1) \cup \mathcal{L}(G_2)$ is finite; but for all G_1, G_2, and the corresponding G, σ can belong to $\{\sigma_G\}^\omega$.

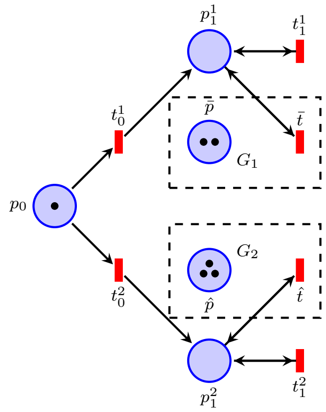

Fig. 2 Sketch for the reduction in the proof of Theorem 2 when $n = 2, 3$, where all transitions outside $G_1 \cup G_2$ are with the same label.

Next we partition the set $\mathcal{R}(N_G, M_0^G)$ of reachable markings of G as follows:

$$\begin{aligned} R_1 =& (\{M \in \mathbb{N}^{P_G} | M(p_0) = 1, M(p_1^1) = M(p_1^2) = 0\} \cup \\ & \{M \in \mathbb{N}^{P_G} | M(p_0) = M(p_1^2) = 0, M(p_1^1) = 1\}) \\ & \cap \mathcal{R}(N_G, M_0^G), \\ R_2 =& \{M \in \mathbb{N}^{P_G} | M(p_0) = M(p_1^1) = 0, M(p_1^2) = 1\} \\ & \cap \mathcal{R}(N_G, M_0^G). \end{aligned} \tag{2}$$

That is, $R_1 \cup R_2 = \mathcal{R}(N_G, M_0^G)$ and $R_1 \cap R_2 = \emptyset$.

If $\mathcal{L}(G_1) \neq \mathcal{L}(G_2)$, without loss of generality, we assume that there exists $\sigma \in \mathcal{L}(G_1) \setminus \mathcal{L}(G_2)$. Then when G generates configuration $\sigma_G \sigma (\sigma_G)^\omega$, it can fire only infinite transition sequences $t_0^1 s (t_1^1)^\omega$, where $s \in (T_1)^*$, $\ell_G(s) = \sigma$. Hence $\emptyset \neq \mathcal{M}(G, \sigma_G \sigma (\sigma_G)^k) \subseteq R_1$ for any $k \in \mathbb{N}$, i.e., G is weakly approximately detectable with respect to 2 and partition (2).

Next we assume that $\mathcal{L}(G_1) = \mathcal{L}(G_2)$. After G started to run, we observed $\sigma_G \sigma_1 \sigma_1' \dots \sigma_k \sigma_k' \sigma$, where $k \in \mathbb{N}$, $\sigma_i \in \Sigma^*$, $\sigma_i' \in \{\sigma_G\}^*$, $\sigma \in \Sigma^* \cup \{\sigma_G\}^*$, $\sigma_1 \dots \sigma_k \in \mathcal{L}(G_1) = \mathcal{L}(G_2)$. If $\sigma \in \Sigma^*$, there exist firing sequences $s_1 \dots s_k s \in (T_1)^*$ of G_1 and $r_1 \dots r_k r \in (T_2)^*$ of G_2 such that $\ell_G(s) = \ell_G(r) = \sigma$, $\ell_G(s_i) = \ell_G(r_i) = \sigma_i$, $i \in [1, k]$. In this case, G fires $t_0^1 s_1 (t_1^1)^{|\sigma_1'|} \dots s_k (t_1^1)^{|\sigma_k'|} s$ and $t_0^2 r_1 (t_1^2)^{|\sigma_1'|} \dots r_k (t_1^2)^{|\sigma_k'|} r$, hence $\mathcal{M}(G, \sigma_G \sigma_1 \sigma_1' \dots \sigma_k \sigma_k' \sigma)$ intersects both R_1 and R_2. If $\sigma \in \{\sigma_G\}^*$, then there exist firing sequences $s_1 \dots s_k \in (T_1)^*$ of G_1 and $r_1 \dots r_k \in (T_2)^*$ of G_2 such that $\ell_G(s_i) = \ell_G(r_i) = \sigma_i$, $i \in [1, k]$. In this case, G fires $t_0^1 s_1 (t_1^1)^{|\sigma_1'|} \dots s_k (t_1^1)^{|\sigma_k' \sigma|}$ and $t_0^2 r_1 (t_1^2)^{|\sigma_1'|} \dots r_k (t_1^2)^{|\sigma_k' \sigma|}$, hence $\mathcal{M}(G, \sigma_G \sigma_1 \sigma_1' \dots \sigma_k \sigma_k' \sigma)$ also intersects both R_1 and R_2. We have checked all possible finite label sequences generated by G, hence G is not weakly approximately detectable with respect to 2 and partition (2).

$n = 3$:

Using the same reduction as the one in the case $n = 2$ and choosing partition

$$\begin{aligned} R_1 =& \{M \in \mathbb{N}^{P_G} | M(p_0) = 1, M(p_1^1) = M(p_1^2) = 0\} \cap \mathcal{R}(N_G, M_0^G), \\ R_2 =& \{M \in \mathbb{N}^{P_G} | M(p_0) = M(p_1^2) = 0, M(p_1^1) = 1\} \cap \mathcal{R}(N_G, M_0^G), \\ R_3 =& \{M \in \mathbb{N}^{P_G} | M(p_0) = M(p_1^1) = 0, M(p_1^2) = 1\} \cap \mathcal{R}(N_G, M_0^G), \end{aligned} \quad (3)$$

similarly we can prove that $\mathcal{L}(G_1) = \mathcal{L}(G_2)$ if and only if G is not weakly approximately detectable with respect to 3 and partition (3).

$n > 3$:

For this case, we construct a more complex reduction. Set $l := n - 1$. Arbitrarily given two ϵ-free labeled P/T nets $G_i = (N_i, M_0^i, \Sigma, \ell_i)$, where $N_i = (P_i, T_i, Pre_i, Post_i)$, $i = 1, 2$, $P_1 \cap P_2 = \emptyset$, $T_1 \cap T_2 = \emptyset$, we next construct a new ϵ-free labeled P/T net G from G_1 and G_2 and a new partition $\{R_1, \ldots, R_{l+1}\}$ of the set of its reachable markings, and prove that $\mathcal{L}(G_1) = \mathcal{L}(G_2)$ if and only if G is not weakly approximately detectable with respect to n and the new partition.

G is specified as $(N_G, M_0^G, \Sigma \cup \{\sigma_G\}, \ell_G)$ (see Fig. 3 as a sketch), where

1. $N_G = (P_G, T_G, Pre_G, Post_G)$;
2. $P_G = P_1 \cup P_2 \cup \{p_0, p_1^1, p_1^2, p_2, \ldots, p_l\}$, $(P_1 \cup P_2) \cap \{p_0, p_1^1, p_1^2, p_2, \ldots, p_l\} = \emptyset$, $|\{p_0, p_1^1, p_1^2, p_2, \ldots, p_l\}| = l + 2$;
3. $T_G = T_1 \cup T_2 \cup \{t_0^1, t_0^2, t_1^1, t_1^2, t_2, \ldots, t_l\}$, $(T_1 \cup T_2) \cap \{t_0^1, t_0^2, t_1^1, t_1^2, t_2, \ldots, t_l\} = \emptyset$, $|\{t_0^1, t_0^2, t_1^1, t_1^2, t_2, \ldots, t_l\}| = l + 3$;
4. $Pre_G|_{P_1 \times T_1} = Pre_1$, $Pre_G|_{P_2 \times T_2} = Pre_2$, $Pre_G(p_0, t_0^1) = Pre_G(p_0, t_0^2) = Pre_G(p_1^1, t_1^1) = Pre_G(p_1^2, t_1^2) = 1$, $Pre_G(p_1^1, t) = 1$ for any $t \in T_1$, $Pre_G(p_1^2, t) = 1$ for any $t \in T_2$, $Pre_G(p_i, t_i) = 1$ for any $i \in [2, l]$, $Pre_G(p, t) = 0$ for any other $(p, t) \in P_G \times T_G$;
5. $Post_G|_{P_1 \times T_1} = Post_1$, $Post_G|_{P_2 \times T_2} = Post_2$, $Post_G(p_1^1, t_0^1) = Post_G(p_1^2, t_0^2) = Post_G(p_2, t_1^1) = Post_G(p_l, t_1^2) = 1$, $Post_G(p_1^1, t) = 1$ for any $t \in T_1$, $Post_G(p_1^2, t) = 1$ for any $t \in T_2$, $Post_G(p_{i+1}, t_i) = 1$ for any $i \in [2, l-1]$, $Post_G(p_2, t_l) = 1$, $Post_G(p, t) = 0$ for any other $(p, t) \in P_G \times T_G$;
6. $M_0^G \in \mathbb{N}^{P_G}$ is specified as $M_0^G|_{P_i} = M_0^i$, $i = 1, 2$, $M_0^G(p_0) = 1$, $M_0^G(p_1^1) = M_0^G(p_1^2) = M_0^G(p_i) = 0$, $i \in [2, l]$;
7. σ_G is a new symbol, i.e., $\sigma_G \notin \Sigma$;
8. $\ell_G|_{T_1} = \ell_1$, $\ell_G|_{T_2} = \ell_2$, $\ell_G(t_0^1) = \ell_G(t_0^2) = \ell_G(t_1^1) = \ell_G(t_1^2) = \ell_G(t_i) = \sigma_G$, $i \in [2, l]$.

For net G, initially only transition t_0^1 or t_0^2 can fire. After t_0^1 (t_0^2) fires, the unique token in place p_0 moves to place p_1^1 (p_1^2), initializing G_1 (G_2). While G_1 (G_2) is running, only transition t_1^1 (t_1^2) outside $T_1 \cup T_2$ can fire. The firing of t_1^1 (t_1^2) moves the token in place p_1^1 (p_1^2) to place p_2 (p_l), and terminates the running of G_1 (G_2), yielding that the token in p_2 (p_l) can move along the direction $p_2 \to \cdots \to p_l \to p_2$ periodically forever, but G_1 (G_2) will never run again. Hence G may fire only infinite transition sequences $t_0^1 s t_1^1 t_2 \ldots t_l t_2 \ldots t_l \ldots$, $t_0^1 s'$, $t_0^2 r t_1^2 t_l t_2 \ldots t_l t_2 \ldots$, or $t_0^2 r'$, where $s \in (T_1)^*$, $s' \in (T_1)^\omega$, $r \in (T_2)^*$, $r' \in (T_2)^\omega$. So G can generate only configurations $\sigma_G \sigma (\sigma_G)^\omega$ or $\sigma_G \sigma'$, where $\sigma \in \Sigma^*$, $\sigma' \in \Sigma^\omega$.

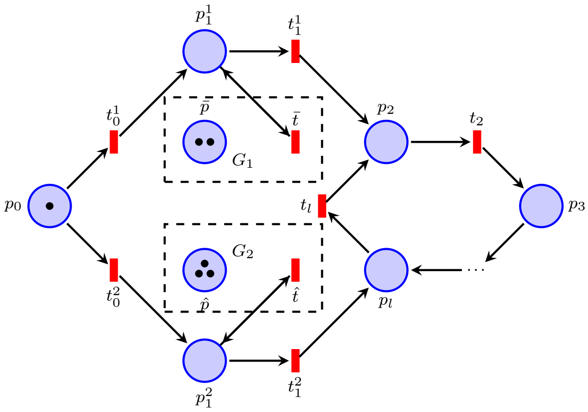

Fig. 3 Sketch for the reduction in the proof of Theorem 2 when $n > 3$, where all transitions outside $G_1 \cup G_2$ are with the same label.

We next partition the set $\mathcal{R}(N_G, M_0^G)$ of reachable markings of G as follows:

$$\begin{aligned} R_1 =& \{M \in \mathbb{N}^{P_G} | M(p_0) \text{ or } M(p_1^1) = 1, \\ & M(p_1^2) = M(p_j) = 0, j \in [2, l]\} \\ & \cap \mathcal{R}(N_G, M_0^G), \\ R_i =& \{M \in \mathbb{N}^{P_G} | M(p_0) = M(p_1^1) = M(p_1^2) = 0, \\ & M(p_i) = 1, M(p_j) = 0, j \in [2, l] \setminus \{i\}\} \\ & \cap \mathcal{R}(N_G, M_0^G), \quad i \in [2, l], \\ R_{l+1} =& \{M \in \mathbb{N}^{P_G} | M(p_1^2) = 1, \\ & M(p_0) = M(p_1^1) = M(p_j) = 0, j \in [2, l]\} \\ & \cap \mathcal{R}(N_G, M_0^G). \end{aligned} \tag{4}$$

That is, $\cup_{i=1}^{l+1} R_i = \mathcal{R}(N_G, M_0^G)$, and $R_i \cap R_j = \emptyset$ for all different $i, j \in [1, l+1]$.

If $\mathcal{L}(G_1) \neq \mathcal{L}(G_2)$, without loss of generality, we assume that there exists $\sigma \in \mathcal{L}(G_1) \setminus \mathcal{L}(G_2)$. Then when G generates configuration $\sigma_G \sigma (\sigma_G)^\omega$, it can fire only transition sequences $t_0^1 s t_1^1 t_2 \dots t_l t_2 \dots t_l \dots$, where $s \in (T_1)^*$, $\ell_G(s) = \sigma$. It can be directly seen for each positive integer k,

$$\emptyset \neq \mathcal{M}(G, \sigma_G \sigma (\sigma_G)^k) \subseteq R_{(k-1) \mod (l-1) + 2},$$

where $(k-1) \mod (l-1)$ means the remainder of $k-1$ divided by $l-1$. That is, G is weakly approximately detectable with respect to n and partition (4).

Next we assume that $\mathcal{L}(G_1) = \mathcal{L}(G_2)$. Note that G generates only configurations $\sigma_G \sigma'$ or $\sigma_G \sigma (\sigma_G)^\omega$, where $\sigma' \in \Sigma^\omega$, $\sigma \in \Sigma^*$. For the former case, for each prefix σ'' of σ', there exist firing sequences $s \in (T_1)^*$ of G_1 and $r \in (T_2)^*$ of G_2 such that $\ell_G(s) = \ell_G(r) = \sigma''$, and markings $M_G, M_G' \in \mathbb{N}^{P_G}$ such that $M_0^G[t_0^1 s\rangle M_G$, $M_0^G[t_0^2 r\rangle M_G'$, $M_G(p_1^1) = 1$, $M_G(p_1^2) = 0$, $M_G'(p_1^1) = 0$, and $M_G'(p_1^2) = 1$, then we have $\mathcal{M}(G,\sigma'')\cap R_1\neq\emptyset$ and $\mathcal{M}(G,\sigma'')\cap R_{l+1}\neq\emptyset$. For the latter case, arbitrarily chosen a prefix $\sigma_G\sigma(\sigma_G)^k$ of $\sigma_G\sigma(\sigma_G)^\omega$, where k is an arbitrary positive integer, we have there exist firing sequences $s\in(T_1)^*$ of G_1 and $r\in(T_2)^*$ of G_2 such that $\ell_G(s)=\ell_G(r)=\sigma$ and G can fire both t_0^1ss' and t_0^2rr', where s' and r' are k length prefixes of $t_2\dots t_lt_2\dots t_l\dots$ and $t_lt_2\dots t_lt_2\dots$, respectively. Since G will fire both t_0^1ss' and t_0^2rr', we have $\mathcal{M}(G,\sigma_G\sigma(\sigma_G)^k)\cap R_{(k-1)\mod (l-1)+2}\neq\emptyset$ and $\mathcal{M}(G,\sigma_G\sigma(\sigma_G)^k)\cap R_{(k-2)\mod (l-1)+2}\neq\emptyset$. Hence for each positive integer k, $\mathcal{M}(G,\sigma_G\sigma(\sigma_G)^k)$ intersects both $R_{(k-1)\mod (l-1)+2}$ and $R_{(k-2)\mod (l-1)+2}$, where $(k-1)\mod (l-1)\neq(k-2)\mod (l-1)$. We have checked all label sequences generated by G, hence G is not weakly approximately detectable with respect to n and partition (4), which completes the proof.

3.2 Instant strong detectability

In this subsection, we study the instant strong detectability of labeled Petri nets.

Definition 3 Consider a labeled P/T net $G=(N=(P,T,Pre,Post),M_0,\Sigma,\ell)$. G is called *instantly strongly detectable* if for each label sequence σ in $\mathcal{L}(G)$, $|\mathcal{M}(G,\sigma)|=1$.

Theorem 3 *1. It is decidable to verify whether a labeled P/T net is instantly strongly detectable.*
2. It is EXPSPACE-hard to check if a labeled P/T net is not instantly strongly detectable in the numbers of places and transitions of the labeled P/T net and the number of tokens of the destination marking in the coverability problem.

Proof To prove decidable result, we reduce the instant strong detectability problem to the home space problem with respect to a computable semi-linear subset.

Given a labeled P/T net $G=(N=(P,T,Pre,Post),M_0,\Sigma,\ell)$, we construct a Petri net $G'=(N'=(P',T',Pre',Post'),M_0')$ which aggregates every two firing sequences of G producing the same label sequence. Denote $P=\{p_1,\dots,p_{|P|}\}$ and $T=\{t_1,\dots,t_{|T|}\}$, duplicate them to $P_i=\{p_1^i,\dots,p_{|P|}^i\}$ and $T_i=\{t_1^i,\dots,t_{|T|}^i\}$, $i=1,2$, where we let $\ell(t_i^1)=\ell(t_i^2)=\ell(t_i)$ for all i in $[1,|T|]$. Then we specify G' as follows[2]:

1. $P'=P_1\cup P_2$;
2. $T'=\{(t_i^1,t_j^2)\in T_1\times T_2|i,j\in[1,|T|],\ell(t_i^1)=\ell(t_j^2)\in\Sigma\}\cup\{(t_1,\epsilon)|t_1\in T_1,\ell(t_1)=\epsilon\}\cup\{(\epsilon,t_2)|t_2\in T_2,\ell(t_2)=\epsilon\}$;
3. for all $k\in[1,2]$, all $l\in[1,|P|]$, and all $i,j\in[1,|T|]$ such that $\ell(t_i^1)=\ell(t_j^2)\in\Sigma$,

$$Pre'(p_l^k,(t_i^1,t_j^2))=\begin{cases}Pre(p_l^k,t_i^1) \text{ if } k=1,\\ Pre(p_l^k,t_j^2) \text{ if } k=2,\end{cases}$$

$$Post'(p_l^k,(t_i^1,t_j^2))=\begin{cases}Post(p_l^k,t_i^1) \text{ if } k=1,\\ Post(p_l^k,t_j^2) \text{ if } k=2;\end{cases}$$

[2] Similar constructions have been used in [28, 29, 33, 32], the differences are that in [33, 32], state pairs producing the same outputs are connected by common inputs, while in [28, 29], transition pairs with the same labels are connected by places.

4. for all $l \in [1, |P|]$, all $i \in [1, |T|]$ such that $\ell(t_i^1) = \ell(t_i^2) = \epsilon$,

$$
\begin{aligned}
Pre'(p_l^1, (t_i^1, \epsilon)) &= Pre(p_l^1, t_i^1),\\
Pre'(p_l^2, (\epsilon, t_i^2)) &= Pre(p_l^2, t_i^2),\\
Post'(p_l^1, (t_i^1, \epsilon)) &= Post(p_l^1, t_i^1),\\
Post'(p_l^2, (\epsilon, t_i^2)) &= Post(p_l^2, t_i^2);
\end{aligned}
$$

5. $M_0'(p_l^k) = M_0(p_l)$ for any k in $[1, 2]$ and any l in $[1, |P|]$.

Assume that there exists a label sequence $\sigma \in \mathcal{L}(G)$ such that $|\mathcal{M}(G, \sigma)| > 1$, then there exist transitions $t_{\mu_1}, \dots, t_{\mu_n}, t_{\omega_1}, \dots, t_{\omega_n} \in T \cup \{\epsilon\}$, where $n \geq 1$, such that $\ell(t_{\mu_1} \dots t_{\mu_n}) = \ell(t_{\omega_1} \dots t_{\omega_n}) = \sigma$, $M_0[t_{\mu_1} \dots t_{\mu_n}\rangle M_1$ and $M_0[t_{\omega_1} \dots t_{\omega_n}\rangle M_2$ for different M_1 and M_2 both in $\mathbb{N}^P$. Then for G', we have $M_0'[(t_{\mu_1}^1, t_{\omega_1}^2) \dots (t_{\mu_n}^1, t_{\omega_n}^2)$ $\rangle M'$, where $M'(p_l^k) = M_k(p_l)$, $k \in [1, 2]$, $l \in [1, |P|]$, and $M'(p_{l'}^1) \neq M'(p_{l'}^2)$ for some $l' \in [1, |P|]$.

On the contrary assume that for each label sequence $\sigma \in \mathcal{L}(G)$, we have $|\mathcal{M}(G, \sigma)| = 1$, then for all $M' \in \mathcal{R}(N', M_0')$, we have $M'(p_l^1) = M'(p_l^2)$ for each l in $[1, |P|]$.

Define

$$
\begin{aligned}
\mathcal{M}_= &= \{M \in \mathbb{N}^{P'} | (\forall l \in [1, |P|])[M(p_l^1) = M(p_l^2)]\},\\
\mathcal{M}_{\neq} &= \{M \in \mathbb{N}^{P'} | (\exists l \in [1, |P|])[M(p_l^1) \neq M(p_l^2)]\}.
\end{aligned}
\tag{5}
$$

Apparently $\mathcal{M}_=$ and $\mathcal{M}_{\neq}$ partition $\mathbb{N}^{P'}$. By the above discussion, G is instantly strongly detectable if and only if $\mathcal{R}(G', M_0') \cap \mathcal{M}_{\neq} = \emptyset$. Then by Proposition 3, we will finish the proof of the decidable result if $\mathcal{M}_{\neq}$ is a computable semi-linear subset of $\mathbb{N}^{P'}$. For each $\bar{P} \subseteq P'$ we define $e_{\bar{P}} \in \mathbb{N}^{P'}$ as

$$
e_{\bar{P}}(p) = \begin{cases} 1 & \text{if } p \in \bar{P},\\ 0 & \text{otherwise.} \end{cases}
$$

Note that $\mathcal{M}_=$ is a linear subset of $\mathbb{N}^{P'}$ with base $\mathbf{0} \in \mathbb{N}^{P'}$ which maps each p in P' to 0 and periods $e_{\{p_l^1, p_l^2\}}$ for all $l \in [1, |P|]$. It is clear that $\mathbb{N}^{P'}$ is a linear subset of itself with based $\mathbf{0}$ and periods $e_{\{p\}}$, where $p \in P'$. Hence by Proposition 2, $\mathcal{M}_{\neq} = \mathbb{N}^{P'} \setminus \mathcal{M}_=$ is a semi-linear subset of $\mathbb{N}^{P'}$ and one can compute its bases and periods from the bases and periods of $\mathbb{N}^{P'}$ and $\mathcal{M}_=$ by using the proof of the proposition, which completes the proof of this part.

Next we prove the hardness result by reducing the coverability problem to the non-instant strong detectability problem in polynomial time.

We are given a Petri net $G = (N = (P, T, Pre, Post), M_0)$ and a destination marking $M \in \mathbb{N}^P$, and construct a labeled P/T net $G' = (N' = (P', T', Pre', Post'),$ $M_0', T \cup \{\sigma_G\}, \ell)$ as follows (see Fig. 4 as a sketch):

1. $P' = P \cup \{p_1, p_2\}$, p_1 and p_2 are different and not in P;
2. $T' = T \cup \{t_1, t_2\}$, t_1 and t_2 are different and not in T;

3. $Pre'|_{P\times T} = Pre$, $Post'|_{P\times T} = Post$, $Pre'(p,t_i) = M(p)$ for each $p \in P$, $Post'(p_i,t_i) = 1$, $i \in [1,2]$, $Pre'(p,t) = Post'(p,t) = 0$ for any other $(p,t) \in P' \times T'$;
4. $\sigma_G \notin T$, $\ell(t) = t$ for each $t \in T$, $\ell(t) = \sigma_G$ for each $t \in \{t_1,t_2\}$;
5. $M'_0|_P = M_0$, $M'_0(p_1) = M'_0(p_2) = 0$.

It is clear that M is not covered by G if and only if neither t_1 nor t_2 fires if and only if G' is instantly strongly detectable. This reduction runs in time linear of the number of places of G and the number of tokens of the destination marking M. Since the coverability problem is EXPSPACE-hard in the number of transitions of G, deciding the instant strong detectability is EXPSPACE-hard in the numbers of places and transitions of G' and the number of tokens of M, which completes the proof.

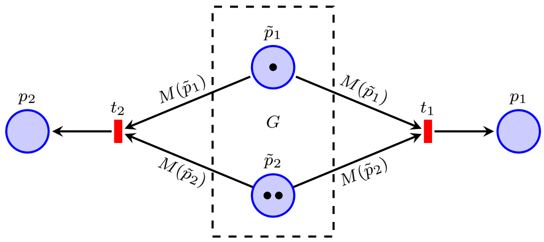

Fig. 4 Sketch for the reduction in the hardness proof of Theorem 3.

Remark 1 The decision algorithm ([2, Corollary 1]) for the home space problem of Petri nets with respect to a semi-linear subset is based on the verification algorithm for the reachability problem of Petri nets [13,9,10]. It was pointed out [3] that the algorithms in [13,9] are non-primitive recursive. And furthermore, the verification algorithm for the home space problem of Petri nets with respect to a semi-linear subset may not be primitive recursive either. It was also pointed out [3] that "Closing the gap between the exponential space lower bound and the non-primitive recursive upper bound is one of the most relevant open problems of net theory.", which actually shows that it is not known whether the instant strong detectability problem of labeled Petri nets belongs to EXPSPACE.

Remark 2 The concept of instant strong detectability of labeled Petri nets is a uniform concept. That is, a labeled Petri net is instantly strongly detectable if and only if it is instantly strongly detectable when its initial marking is replaced by any of its reachable markings. Formally, for a labeled Petri net $G = (N, M_0, \Sigma, \ell)$, G is instantly strongly detectable if and only if $G' = (N, M, \Sigma, \ell)$ is instantly strongly detectable for each $M \in \mathcal{R}(N, M_0)$. The sufficiency naturally holds since $M_0 \in \mathcal{R}(N, M_0)$. For the necessity, if there exists $M_1 \in \mathcal{R}(N, M_0)$ such that labeled Petri net $G_1 = (N, M_1, \Sigma, \ell)$ is not instantly strongly detectable, then there exists $\sigma_1 \in \mathcal{L}(G_1)$ satisfying $|\mathcal{M}(G_1, \sigma_1)| > 1$. Since there exists $\sigma_0 \in \mathcal{L}(G)$ satisfying $M_1 \in \mathcal{M}(G, \sigma_0)$, we have $\mathcal{M}(G, \sigma_0\sigma_1) \supset \mathcal{M}(G_1, \sigma_1)$ and $|\mathcal{M}(G, \sigma_0\sigma_1)| > 1$, i.e., G is not instantly strongly detectable. Hence if a labeled Petri net is instantly strongly detectable, in order to determine the current marking, one does not need to care about when the net started to run.

4 Conclusion

In this paper, we proved that the problems of verifying weak detectability of labeled Petri nets with inhibitor arcs and weak approximate detectability of labeled Petri nets are both undecidable. We also proved that the problem of verifying instant strong detectability of labeled Petri nets is decidable, and its inverse problem is EXPSPACE-hard in the number of transitions, where the instant strong detectability means whether each label sequence generated by the labeled Petri net can be used to reconstruct the current marking. It is not difficult to obtain that all these problem are decidable for bounded labeled Petri nets, so it is interesting to design fast verification algorithms for them.

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