Analysis of Stochastic Switched Systems with Application to Networked Control Under Jamming Attacks

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***Abstract*—We investigate the stability problem for discrete-time stochastic switched linear systems under the specific scenarios where information about the switching patterns and the probability of switches are not available. Our analysis focuses on the average number of times each mode becomes active in the long run and, in particular, utilizes their lower- and upper-bounds. This setup is motivated by cyber security issues for networked control systems in the presence of packet losses due to malicious jamming attacks where the attacker's strategy is not known a priori. We derive a sufficient condition for almost sure asymptotic stability of the switched systems which can be examined by solving a linear programming problem. Our approach exploits the dynamics of an equivalent system that describes the evolution of the switched system's state at every few steps; the stability analysis may become less conservative by increasing the step size. The computational efficiency is further enhanced by exploiting the structure in the stability analysis problem, and we introduce an alternative linear programming problem that has fewer variables. We demonstrate the efficacy of our results by analyzing networked control problems where communication channels face random packet losses as well as jamming attacks.**

***Index Terms*—Stochastic switched systems, stability analysis, linear programming, networked control systems, jamming, packet losses.**

I. Introduction

In the recent studies of hybrid systems, switched systems represent an important and fundamental class due to their simple structures. Switched systems are composed of a number of subsystems that possess different continuous dynamics and a discrete-valued switching mode signal which determines the active subsystem. Complicated behaviors in the state evolutions can be demonstrated, depending on the nature of the switching as well as the dynamics of the subsystems. For this reason, analyses of switched systems concerning their stability and performance have posed various challenges and led researchers to interesting results [1], [2].

Recently, the importance of this class of hybrid systems is rising from the application side. In particular, in large-scale networked control systems, switchings in the system dynamics frequently take place whenever the status in the communication networks changes. In such systems, network channels connect the plant having many sensors and actuators with remote controllers. Thus, transmission times and patterns of control-related signals affect the dynamics of the plant as well as the overall closed-loop system. Furthermore, the communication is often unreliable in that the transmitted data may become lost, not reaching the destination depending on the condition of the channels. Since such channel behaviors are commonly modeled in a probabilistic manner, the study of networked control systems often call for the framework of stochastic switched systems.

Stability of switched systems has been studied for different types of mode signals. The case of (deterministic) arbitrary switching has been explored in a number of works including [3], [4], establishing conditions under which a switched system remains stable for all possible mode switching scenarios. The switching frequency in the mode signal can be restricted by utilizing dwell-time and average dwell-time notions. The works [5], [6] dealt with systems composed of only stable subsystems while [7], [8] made extensions to systems also consisting of unstable subsystems. In addition, [9] investigated the problem of designing state-dependent switching rules to guarantee stability.

For stochastic mode signals, stability issues of switched systems have attracted considerable attention as well (see [10], [11] and the references therein). In most cases, stability results for such systems rely on statistical information on the mode signal such as the probability of mode switches and the stationary distributions associated with the modes. An important class there is that of Markov jump systems, for which the mode signal is dominated by underlying Markov chains [12]–[16]. Moreover, some works characterized both stochastic and deterministic effects in the switching of the mode signal, referred to as "dual switching" [17], [18].

In this paper, we investigate a stability problem for discrete-time stochastic switched linear systems with a special emphasis on the case where information about the mode switching probabilities or the stationary distributions are not available for analysis. Our interest stems from the current research activities on cyber security of networked control systems [19]–[21]. Today, more control systems are connected to the Internet and wireless networks for their remote operation and monitoring. Such communication settings significantly increase the risk to be targeted by malicious cyber attackers. Clearly, the system dynamics can change, for example, if attackers interfere with the communication of the control related signals. Under such conditions, the networked system may be represented as a stochastic switched system, but the a priori knowledge on the switching, whether it is deterministic or probabilistic, would be extremely limited. In the networked control literature, recent works dealt with denial-of-service (DoS) and jamming attacks [22]–[24] or packet drops by compromised routers [25], [26].

Our problem formulation is based on the approach in our

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recent work [21]; see also [27]–[30] for more related results. There, we focused on the stochastic stability of a networked control system under malicious jamming attacks, where the jamming causes losses of data during packet transmissions due to strong interferences. The normal operation and the dynamics under jamming attacks are represented with two different modes. The transmission failure instants and the probability of transmission failures are influenced by the attacker's actions and hence not available for analysis. In [21], we introduced a novel stochastic model regarding the timings of jamming using only the asymptotic tail probabilities of average transmission failures. Roughly speaking, it corresponds to the bound on the average ratio of time that the communication is blocked due to jamming. A key property of the model lies in its generality, being capable to describe both malicious and nonmalicious data losses. Specifically, it can represent discrete-time versions of the jamming and denial-of-service attack models in [19], [20] as well as random packet loss models based on Bernoulli and Markov processes [31]–[34].

In this paper, as a generalization of this class of systems in networked control under jamming attacks, we consider stochastic switched systems having arbitrary number of modes. The limited knowledge on the model signal is represented by the lower- and upper-bounds on the percentage of the time each mode is active in the long run. We pay special attention to improving the conservatism and computational efficiency for the stability analysis. It is established that linear programming can be used for identifying the worst mode switching scenario in terms of stability, and we assess the stability of the system under that scenario. This stability assessment approach is different from the matrix inequality-type stability conditions obtained with the Lyapunov-based stability analysis in [21]. Certain aspects are generalized in the linear programming-based conditions and for this reason less conservative analysis is possible. In this paper, we will discuss these points in detail and also illustrate by numerical examples. Moreover, the proposed approach differs from those in [35]–[38], where linear programming is used for constructing Lyapunov functions.

In the course of our analysis, we first construct a new system that describes the evolution of the switched system's state at every $h \in \mathbb{N}$ steps. For a given switched system with M modes, this new "lifted" version of the system is composed of M^h modes, each of which is identified by a sequence of h numbers indicating a progression of modes in the original system. The advantage of the approach is that the conservatism in the analysis may be reduced by increasing the size of h. This is possible because, with large h, stability/instability properties of more switching patterns are taken into account. Furthermore, to reduce the computational complexity, we exploit the problem structure and develop an alternative linear programming problem. This problem shares the same optimal objective function value with the original problem, but involves fewer variables, which are in fact polynomial with respect to the problem size.

The idea of studying a new system that describes the state's evolution at every few steps has previously been employed in [39]–[42]. The works [39] and [40] explore Markov jump systems, but assume the full knowledge of the Markov chain. On the other hand, in [41] and [42], stability problems under constrained switching rules are addressed, where all possible switching patterns are identified through graphs that indicate the constraints in the switching. In our problem setting, information about the possible switching patterns is not available, and moreover, we consider scenarios where switching is allowed between all pairs of modes. For this particular setting, the stability results in [41], [42] require all subsystems to be stable. Therefore, they are not applicable for our problem since we allow both stable and unstable subsystem dynamics. In addition to the analysis of switched systems, an approach similar to the investigation of the dynamics over multiple time steps has also been used for establishing the convergence of risk-sensitive and risk sensitive like filters in [43], [44].

We show in the paper that our results are applicable to a wide class of mode signals. In particular, it includes those that are the outputs of hidden Markov chains whose actual state space and transition probabilities are unknown. In networked control problems under random and malicious packet transmission failures such signals may arise in periodic attacks discussed in [19] as well as random packet losses on channels described with Markov models studied in [45], [46]. We apply our stability results in two networked control problems, where the plant and the remotely located controller exchange information packets over one or more channels subject to packet losses due to random communication errors or malicious jamming attacks.

The rest of the paper is organized as follows. In Section II, we explain the switched system dynamics and obtain sufficient stability conditions by utilizing bounds on the average number of times each mode is active in the long run. In Section III, we present our approach for checking stability conditions by means of solving linear programming problems. Moreover, in Section IV, we discuss the application of our results in the problem of networked control under jamming attacks. We present numerical examples in Section V. Finally, in Section VI, we conclude the paper.

We note that part of the results in Sections II and III appeared in our preliminary report [47] for the special case of two modes in the context of a networked control problem. Here, we provide complete proofs and more detailed discussions for the general modes case.

We use a fairly standard notation in the paper. Specifically, $\mathbb{N}$ and $\mathbb{N}_0$ respectively denote positive and nonnegative integers. A finite-length sequence of ordered elements $q_1, q_2, \ldots, q_h$ is represented by $q = (q_1, q_2, \ldots, q_h)$. We use $\lfloor \cdot \rfloor$ to denote the largest integer that is smaller than or equal to its real argument. The notation $\mathbb{P}[\cdot]$ denotes the probability on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, we utilize $\mathbb{1}[E] : \Omega \to \{0, 1\}$ for the indicator of the event $E \in \mathcal{F}$, that is, $\mathbb{1}[E](\omega) = 1$ for $\omega \in E$, and $\mathbb{1}[E](\omega) = 0$ for $\omega \notin E$.

II. Stochastic Switched System Stability Analysis

In this section we first describe the dynamics of a stochastic switched system. We then discuss the stability problem and provide sufficient almost sure asymptotic stability conditions.

A. Switched System Dynamics

Consider the discrete-time switched linear system with $M \in \mathbb{N}$ modes described by

$$x(t+1) = A_{r(t)}x(t), \quad x(0) = x_0, r(0) = r_0, t \in \mathbb{N}_0, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $\{r(t) \in \{1, \ldots, M\}\}_{t\in\mathbb{N}_0}$ is the mode signal, and $A_s \in \mathbb{R}^{n\times n}$, $s \in \{1, \ldots, M\}$, represent the system matrices for each mode. We use $\mathcal{M} \triangleq \{1, \ldots, M\}$ to denote the set of modes.

The mode signal $\{r(t) \in \mathcal{M}\}_{t\in\mathbb{N}_0}$ is assumed to be a stochastic process that satisfies the following assumption.

Assumption 2.1: There exist scalars $\underline{\rho}_s, \overline{\rho}_s \in [0, 1]$, $s \in \mathcal{M}$, such that

$$\liminf_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[r(t) = s] \geq \underline{\rho}_s, \quad (2)$$

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[r(t) = s] \leq \overline{\rho}_s, \quad (3)$$

almost surely.

In Assumption 2.1, the scalars $\underline{\rho}_s$ and $\overline{\rho}_s$ respectively represent lower- and upper-bounds on the long-run average number of times mode s is active. If no information is available on the long-run average for mode s, the scalars $\underline{\rho}_s$ and $\overline{\rho}_s$ can be selected as $\underline{\rho}_s = 0$ and $\overline{\rho}_s = 1$, since (2) and (3) are trivially satisfied with those values for all $s \in \mathcal{M}$.

Our main motivation for considering Assumption 2.1 is to analyze switched systems for which precise information about the mode switching rules is not available for analysis. In Section II-B, we show that the scalars $\underline{\rho}_s$, $\overline{\rho}_s$, $s \in \mathcal{M}$, can be used for analysis, even if we do not know how the mode may switch at each time t.

Assumption 2.1 allows the mode signal $\{r(t) \in \mathcal{M}\}_{t\in\mathbb{N}_0}$ to be generated in many different ways either randomly according to a probability distribution or in a deterministic fashion. For instance, $\{r(t) \in \mathcal{M}\}_{t\in\mathbb{N}_0}$ may be a stationary and ergodic stochastic process with stationary distribution $\pi \in [0, 1]^M$. In that case $\underline{\rho}_s$ and $\overline{\rho}_s$ would be scalars that satisfy $\underline{\rho}_s \leq \pi_s \leq \overline{\rho}_s$, $s \in \mathcal{M}$. On the other hand, $\{r(t) \in \mathcal{M}\}_{t\in\mathbb{N}_0}$ may also represent a deterministically generated switching sequence. For example, a particular mode switching pattern of length $T \in \mathbb{N}$ can be repeated to create a periodic switching sequence. In that case, $\underline{\rho}_s$ and $\overline{\rho}_s$ would correspond to lower- and upper-bounds on the ratio of the number of times mode s is active in the T-length switching pattern. In the case where $\{r(t) \in \mathcal{M}\}_{t\in\mathbb{N}_0}$ is deterministic, the limits $\lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[r(t) = q]$, $q \in \mathcal{M}$, when they exist, correspond to the "discrete event rate" utilized by [48], [49] for deterministic systems.

Remark 2.1: In the literature of stochastic switched systems the mode signal is typically characterized as a Markov chain [13]. In this paper, $\{r(t) \in \mathcal{M}\}_{t\in\mathbb{N}_0}$ is not necessarily a Markov chain. In fact in certain cases, we may have

$$\mathbb{P}[r(t+1) = q | r(0), r(1), \ldots, r(t)] \neq \mathbb{P}[r(t+1) = q | r(t)],$$

which indicates that $r(\cdot)$ fails to satisfy the Markov property [50]. Furthermore, $r(\cdot)$ may also depend on other processes. Our main motivation for considering this general setup for the mode signal $\{r(t) \in \mathcal{M}\}_{t\in\mathbb{N}_0}$ comes from the networked control problem under jamming attacks. As we illustrate in Section IV, a networked control system under jamming attacks can be represented by the switched system (1). The mode signal of such a switched system is not necessarily a Markov process, since the attacker can influence the mode signal in various ways. Consequently, the analysis of the switched system (1) requires methods that are different from those utilized in switched systems with Markov modes [13].

In our stability analysis in the following section, we utilize the scalars $\underline{\rho}_s$ and $\overline{\rho}_s$ satisfying (2) and (3) instead of transition probabilities or switching patterns. In Section IV, we model networked control systems as switched systems, where the mode signals represent the state of certain communication channels that face jamming attacks. There, $\underline{\rho}_s$ and $\overline{\rho}_s$ are characterized based on the information about the average number of times transmissions fail due to attacks in the long run.

B. Stability Analysis

In this section, we explore the stability of the switched system (1), where the mode signal satisfies Assumption 2.1. We use the stochastic stability notion of *almost sure asymptotic stability* in our analysis.

Definition 2.2: The zero solution $x(t) \equiv 0$ of the stochastic system (1) is *almost surely stable* if for each $\epsilon > 0$ and $\bar{p} > 0$, there exists $\delta = \delta(\epsilon, \bar{p}) > 0$ such that if $\|x_0\|_2 < \delta$, then

$$\mathbb{P}[\sup_{t\in\mathbb{N}_0} \|x(t)\|_2 > \epsilon] < \bar{p}, \quad (4)$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Moreover, the zero solution $x(t) \equiv 0$ is *asymptotically stable almost surely* if it is almost surely stable and

$$\mathbb{P}[\lim_{t\to\infty} \|x(t)\|_2 = 0] = 1. \quad (5)$$

Stability of discrete-time switched systems has been investigated in many studies under different assumptions on the mode signal. For instance, in several works (see [2] and the references therein) researchers explore stability of systems with a form similar to (1) under arbitrary switching. A necessary condition is stability of each mode (i.e., $A_1, A_2, \ldots, A_M$ need to be Schur matrices). In our problem setting we allow some of the modes to be unstable, and hence this approach is not applicable.

On the other hand, researchers also studied stability of systems similar to (1) for the case where $r(\cdot)$ is a Markov process (see, e.g., [12], [13], [39]). The stability analysis in those studies relies on transition probabilities and stationary distributions associated with the Markov process that characterize the switching sequence. Note again that in our case, $r(\cdot)$ need not be a Markov process. Furthermore, to account for the uncertainty in generation of the mode signal, we assume that statistical information concerning transition probabilities and stationary distributions is not available. Hence, the stability results reported in the above-mentioned literature are not applicable to the present problem.

In our stability analysis of the switched system, we follow the approach in [39], [41] and investigate the evolution of the system's state at every $h \in \mathbb{N}$ steps. First, let $\mathcal{M}^h$ denote the set of sequences of length h with entries in $\mathcal{M}$, that is,

$$\mathcal{M}^h \triangleq \{(q_1, q_2, \ldots, q_h) \colon q_j \in \mathcal{M}, j \in \{1, \ldots, h\}\}.$$

With this definition, q_i (ith entry of a sequence q) represents a mode in the set of modes $\mathcal{M}$. Now, let $\{\bar{r}(i) \in \mathcal{M}^h\}_{i\in\mathbb{N}_0}$ be a *sequence-valued* process defined by

$$\bar{r}(i) \triangleq (r(ih), r(ih+1), \ldots, r((i+1)h-1)), \;\; i \in \mathbb{N}_0. \quad (6)$$

It then follows that the state evaluated at every h steps is described by

$$x((i+1)h) = \Gamma_{\bar{r}(i)} x(ih), \quad i \in \mathbb{N}_0, \quad (7)$$

where

$$\Gamma_q \triangleq A_{q_h} A_{q_{h-1}} \cdots A_{q_1}, \quad q \in \mathcal{M}^h. \quad (8)$$

The dynamical system (7) is a "lifted" switched system with M^h number of modes. Each mode of this system is identified by a sequence of h numbers from $\mathcal{M}$ representing the modes of the original switched system (1).

Now, let $c_s \colon \mathcal{M}^h \to \{0, \ldots, h\}$ be defined by $c_s(q) \triangleq \sum_{j=1}^{h} \mathbb{1}[q_j = s], q \in \mathcal{M}^h, s \in \mathcal{M}$. With this definition, the number of entries with value s in the sequence $q \in \mathcal{M}^h$ is represented with $c_s(q)$. Note that c_s satisfies

$$\sum_{i=0}^{k-1} \sum_{q\in\mathcal{M}^h} c_s(q) \mathbb{1}[\bar{r}(i) = q] = \sum_{i=0}^{kh-1} \mathbb{1}[r(i) = s], \;\; k \in \mathbb{N}, \quad (9)$$

which establishes a key relation between the mode signal $r(\cdot)$ and the sequence-valued process $\bar{r}(\cdot)$.

In Lemma 2.3 below, we use (9) to obtain a relation between $\underline{\rho}_s, \overline{\rho}_s$ in Assumption 2.1 and the long-run average numbers of the occurrences of all sequences in $\mathcal{M}^h$. The long run average for a sequence $q \in \mathcal{M}^h$ is given by

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q],$$

whenever this limit exists, that is, $\frac{1}{k}\sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ converges almost surely to a random variable as $k \to \infty$.

Lemma 2.3: Suppose $\{r(t) \in \mathcal{M}\}_{t\in\mathbb{N}_0}$ satisfies Assumption 2.1 with $\underline{\rho}_s, \overline{\rho}_s \in [0, 1]$, $s \in \mathcal{M}$. If $\lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ exists for each $q \in \mathcal{M}^h$, then we have

$$\sum_{q\in\mathcal{M}^h} \frac{c_s(q)}{h} \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] \geq \underline{\rho}_s, \quad (10)$$

$$\sum_{q\in\mathcal{M}^h} \frac{c_s(q)}{h} \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] \leq \overline{\rho}_s, \quad (11)$$

for $s \in \mathcal{M}$, almost surely.

Proof: We first show (11). By (9),

$$\begin{aligned} &\sum_{q\in\mathcal{M}^h} \frac{c_s(q)}{h} \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] \\ &\quad = \lim_{k\to\infty} \frac{1}{kh} \sum_{i=0}^{k-1} \sum_{q\in\mathcal{M}^h} c_s(q) \mathbb{1}[\bar{r}(i) = q] \\ &\quad = \lim_{k\to\infty} \frac{1}{kh} \sum_{i=0}^{kh-1} \mathbb{1}[r(i) = s]. \end{aligned} \quad (12)$$

Here, we have

$$\begin{aligned} &\Big\{\frac{1}{\bar{k}h} \sum_{i=0}^{\bar{k}h-1} \mathbb{1}[r(i) = s] \colon \bar{k} \geq k\Big\} \\ &\quad \subseteq \Big\{\frac{1}{\bar{k}} \sum_{i=0}^{\bar{k}-1} \mathbb{1}[r(i) = s] \colon \bar{k} \geq k\Big\}, \quad k \in \mathbb{N}, \end{aligned} \quad (13)$$

and hence

$$\sup_{\bar{k}\geq k} \frac{1}{\bar{k}h} \sum_{i=0}^{\bar{k}h-1} \mathbb{1}[r(i) = s] \leq \sup_{\bar{k}\geq k} \frac{1}{\bar{k}} \sum_{i=0}^{\bar{k}-1} \mathbb{1}[r(i) = s], \quad k \in \mathbb{N}. \quad (14)$$

Therefore,

$$\limsup_{k\to\infty} \frac{1}{kh} \sum_{i=0}^{kh-1} \mathbb{1}[r(i) = s] \leq \limsup_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[r(i) = s]. \quad (15)$$

As a result, (11) follows from (3), (12), and (15).

The inequality (10) can be shown using a similar approach. □

Next, we employ Lemma 2.3 to establish sufficient conditions for almost sure asymptotic stability. To this end, first, for a given matrix $N \in \mathbb{R}^{n\times n}$, let $\|N\|$ denote the induced matrix norm defined by

$$\|N\| \triangleq \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\|Nx\|}{\|x\|}, \quad (16)$$

where $\|\cdot\|$ on the right-hand side denotes a vector norm on $\mathbb{R}^n$. In the proof of the next result, we use the submultiplicativity property of induced matrix norms, i.e., $\|N_1 N_2\| \leq \|N_1\|\|N_2\|$ for $N_1, N_2 \in \mathbb{R}^{n\times n}$ (see Section 5.6 in [51]).

Theorem 2.4: Consider the switched system (1). Suppose that the mode signal $\{r(t) \in \{1, \ldots, M\}\}_{t\in\mathbb{N}_0}$ satisfies Assumption 2.1 with $\overline{\rho}_s, \underline{\rho}_s \in [0, 1]$, $s \in \mathcal{M}$, and $\lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ exists for each $q \in \mathcal{M}^h$ for a given $h \in \mathbb{N}$. If there exist an induced matrix norm $\|\cdot\|$ and a scalar $\varepsilon \in (0, 1)$ such that the inequality

$$\sum_{q\in\mathcal{M}^h} \gamma_q \rho_q < 0, \quad (17)$$

holds with

$$\gamma_q \triangleq \begin{cases} \ln \|\Gamma_q\|, & \Gamma_q \neq 0, \\ \ln \varepsilon, & \Gamma_q = 0, \end{cases} \quad q \in \mathcal{M}^h, \quad (18)$$

for all $\rho_q \in [0,1], q \in \mathcal{M}^h$, that satisfy

$$\sum_{q\in\mathcal{M}^h} \rho_q = 1, \tag{19}$$

$$\underline{\rho}_s \le \sum_{q\in\mathcal{M}^h} \frac{c_s(q)}{h}\rho_q \le \overline{\rho}_s, \quad s \in \mathcal{M}, \tag{20}$$

then the zero solution $x(t) \equiv 0$ of the dynamical system (1) is asymptotically stable almost surely.

Proof: First, it follows from (7) that

$$\|x((k+1)h)\| = \|\Gamma_{\bar{r}(k)}x(kh)\| \le \|\Gamma_{\bar{r}(k)}\|\|x(kh)\|,$$

and hence by the submultiplicativity property of the induced matrix norm $\|\cdot\|$, we have

$$\|x(kh)\| \le \eta(k)\|x_0\|, \quad k \in \mathbb{N}_0, \tag{21}$$

where $\eta(k) \triangleq \prod_{i=0}^{k-1}\|\Gamma_{\bar{r}(i)}\|$. Now we define $\mu(k) \triangleq \sum_{i=0}^{k-1}\gamma_{\bar{r}(i)}$, $k \in \mathbb{N}_0$, where $\gamma_q, q \in \mathcal{M}^h$, are given by (18). It follows from (18) together with the definitions of $\eta(k)$ and $\mu(k)$ that $\eta(k) \le e^{\mu(k)}, k \in \mathbb{N}_0$. Furthermore, since $\gamma_{\bar{r}(i)} = \sum_{q\in\mathcal{M}^h}\gamma_q \mathbb{1}[\bar{r}(i) = q]$, we have

$$\mu(k) = \sum_{i=0}^{k-1}\sum_{q\in\mathcal{M}^h}\gamma_q\mathbb{1}[\bar{r}(i) = q] = \sum_{q\in\mathcal{M}^h}\gamma_q\sum_{i=0}^{k-1}\mathbb{1}[\bar{r}(i) = q],$$

for $k \in \mathbb{N}$, and as a result,

$$\lim_{k\to\infty}\frac{1}{k}\mu(k) = \sum_{q\in\mathcal{M}^h}\gamma_q\lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\mathbb{1}[\bar{r}(i) = q], \tag{22}$$

almost surely. Here, note that $\lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\mathbb{1}[\bar{r}(i) = q] \in [0,1]$. Furthermore,

$$\begin{aligned}&\sum_{q\in\mathcal{M}^h}\lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\mathbb{1}[\bar{r}(i) = q]\\ &\quad= \lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\sum_{q\in\mathcal{M}^h}\mathbb{1}[\bar{r}(i) = q] = \lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}1 = 1.\end{aligned} \tag{23}$$

Let $\rho_q^* \triangleq \lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$, and

$$\vartheta \triangleq \max\Big\{\sum_{q\in\mathcal{M}^h}\gamma_q\rho_q\colon \rho_q \in [0,1], q \in \mathcal{M}^h, \text{ s.t. } (19), (20)\Big\}.$$

Furthermore, let $E, F \subset \mathcal{F}$ be the events defined by

$$\begin{aligned}E \triangleq \{\omega \in \Omega\colon &\sum_{q\in\mathcal{M}^h}\rho_q^* = 1,\\ &\underline{\rho}_s \le \sum_{q\in\mathcal{M}^h}\frac{c_s(q)}{h}\rho_q^* \le \overline{\rho}_s,\ s \in \mathcal{M}\}.\\ F \triangleq \{\omega \in \Omega\colon &\sum_{q\in\mathcal{M}^h}\gamma_q\rho_q^* \le \vartheta\}.\end{aligned}$$

We first show that $\mathbb{P}[E] = 1$. Observe that (23) implies (19) with ρ_q replaced by ρ_q^*. Moreover, Lemma 2.3 implies that (20) with ρ_q replaced by ρ_q^* holds almost surely. Hence, we have $\mathbb{P}[E] = 1$. Now, since $E \subseteq F$, we also have $\mathbb{P}[F] = 1$. Furthermore, (17) implies that $\vartheta < 0$. As a result, by noting that $\mathbb{P}[F] = 1$, we obtain $\sum_{q\in\mathcal{M}^h}\gamma_q\rho_q^* \le \vartheta < 0$, almost surely. We use this fact together with (22) to obtain

$$\lim_{k\to\infty}\frac{1}{k}\mu(k) = \sum_{q\in\mathcal{M}^h}\gamma_q\rho_q^* \le \vartheta < 0, \tag{24}$$

almost surely. Here, (24) implies $\lim_{k\to\infty}\mu(k) = -\infty$, almost surely. As a result, by noting that $\eta(k) \le e^{\mu(k)}$, we obtain $\mathbb{P}[\lim_{k\to\infty}\eta(k) = 0] = 1$. Thus, for any $\epsilon > 0$, $\lim_{j\to\infty}\mathbb{P}[\sup_{k\ge j}\eta(k) > \epsilon] = 0$ (see Proposition 5.6 of [52]). Therefore, for any $\epsilon > 0$ and $\bar{p} > 0$, there exists a positive integer $N(\epsilon, \bar{p})$ such that

$$\mathbb{P}[\sup_{k\ge j}\eta(k) > \epsilon] < \bar{p}, \quad j \ge N(\epsilon, \bar{p}). \tag{25}$$

In what follows, we show almost sure stability of the switched system by using (21) and (25). First, we define

$$\phi \triangleq \max\{1, \max_{s\in\mathcal{M}}\|A_s\|\} \tag{26}$$

and $\mathcal{T}_k \triangleq \{kh, \ldots, (k+1)h - 1\}$, $k \in \mathbb{N}_0$. Using these definitions, we obtain

$$\begin{aligned}\|x(t+1)\| &= \|A_{r(t)}x(t)\| \le \|A_{r(t)}\|\|x(t)\|\\ &\le \phi\|x(t)\|, \quad t \in \mathcal{T}_k.\end{aligned} \tag{27}$$

It then follows from (27) that $\|x(t)\| \le \phi^{t-kh}\|x(kh)\|, t \in \mathcal{T}_k$. Since $\mathcal{T}_k$ has h time instants and $\phi \ge 1$, we have $\phi^{t-kh} \le \phi^{h-1} \le \phi^h$ and hence $\|x(t)\| \le \phi^h\|x(kh)\|$ for all $t \in \mathcal{T}_k$. Consequently,

$$\max_{t\in\mathcal{T}_k}\|x(t)\| \le \phi^h\|x(kh)\|, \quad k \in \mathbb{N}_0. \tag{28}$$

Now by (21) and (28),

$$\eta(k) \ge \|x(kh)\|\|x_0\|^{-1} \ge \max_{t\in\mathcal{T}_k}\|x(t)\|\phi^{-h}\|x_0\|^{-1},\ k \in \mathbb{N}_0.$$

Then it follows from (25) that for all $\epsilon > 0$ and $\bar{p} > 0$,

$$\begin{aligned}&\mathbb{P}[\sup_{k\ge j}\max_{t\in\mathcal{T}_k}\|x(t)\| > \epsilon\phi^h\|x_0\|]\\ &\quad= \mathbb{P}[\sup_{k\ge j}\max_{t\in\mathcal{T}_k}\|x(t)\|\phi^{-h}\|x_0\|^{-1} > \epsilon]\\ &\quad\le \mathbb{P}[\sup_{k\ge j}\eta(k) > \epsilon] < \bar{p}, \quad j \ge N(\epsilon, \bar{p}).\end{aligned}$$

Now let $\delta_1 \triangleq \phi^{-h}$. Notice that if $\|x_0\| \le \delta_1$, then $\phi^h\|x_0\| \le 1$, and therefore, for all $j \ge N(\epsilon, \bar{p})$, we have

$$\begin{aligned}&\mathbb{P}[\sup_{k\ge j}\max_{t\in\mathcal{T}_k}\|x(t)\| > \epsilon]\\ &\quad\le \mathbb{P}[\sup_{k\ge j}\max_{t\in\mathcal{T}_k}\|x(t)\| > \epsilon\phi^h\|x_0\|] < \bar{p}.\end{aligned} \tag{29}$$

Furthermore, observe that for all $k \in \{0, 1, \ldots, N(\epsilon, \bar{p}) - 1\}$, we have $\|x(kh)\| \le \phi^k\|x_0\| \le \phi^{N(\epsilon,\bar{p})-1}\|x_0\|$. Hence, as a result of (28),

$$\max_{t\in\mathcal{T}_k}\|x(t)\| \le \phi^h\|x(kh)\| \le \phi^{h+N(\epsilon,\bar{p})-1}\|x_0\|, \tag{30}$$

for all $k \in \{0, 1, \ldots, N(\epsilon, \bar{p}) - 1\}$. Let $\delta_2(\epsilon, \bar{p}) \triangleq \epsilon\phi^{1-h-N(\epsilon,\bar{p})}$. Now, if $\|x_0\| \le \delta_2(\epsilon, \bar{p})$, then by (30),

$\max_{t\in\mathcal{T}_k}\|x(t)\| \leq \epsilon$, $k \in \{0,1,\ldots,N(\epsilon,\bar{p})-1\}$. Thus, if $\|x_0\| \leq \delta_2(\epsilon,\bar{p})$, then

$$\mathbb{P}[\max_{k\in\{0,1,\ldots,N(\epsilon,\bar{p})\}}\max_{t\in\mathcal{T}_k}\|x(t)\| > \epsilon] = 0. \tag{31}$$

Due to (29) and (31), for all $\epsilon > 0$, $\bar{p} > 0$, we have

$$\begin{aligned}\mathbb{P}[\sup_{t\in\mathbb{N}_0}\|x(t)\| > \epsilon] &= \mathbb{P}[\sup_{k\in\mathbb{N}_0}\max_{t\in\mathcal{T}_k}\|x(t)\| > \epsilon]\\ &= \mathbb{P}[\{\max_{k\in\{0,\ldots,N(\epsilon,\bar{p})-1\}}\max_{t\in\mathcal{T}_k}\|x(t)\| > \epsilon\}\\ &\qquad \cup \{\sup_{k\geq N(\epsilon,\bar{p})}\max_{t\in\mathcal{T}_k}\|x(t)\| > \epsilon\}]\\ &\leq \mathbb{P}[\max_{k\in\{0,\ldots,N(\epsilon,\bar{p})-1\}}\max_{t\in\mathcal{T}_k}\|x(t)\| > \epsilon]\\ &\quad + \mathbb{P}[\sup_{k\geq N(\epsilon,\bar{p})}\max_{t\in\mathcal{T}_k}\|x(t)\| > \epsilon] < \bar{p},\end{aligned} \tag{32}$$

whenever $\|x_0\| < \min(\delta_1, \delta_2(\epsilon,\bar{p}))$.

Now, by Corollary 5.4.5 of [51], there exist $c_1, c_2 > 0$ such that

$$c_1\|x\| \leq \|x\|_2 \leq c_2\|x\|, \quad x \in \mathbb{R}^n. \tag{33}$$

By (32) and (33), we obtain that for all $\epsilon > 0$, $\bar{p} > 0$,

$$\mathbb{P}[\sup_{t\in\mathbb{N}_0}\|x(t)\|_2 > \epsilon] \leq \mathbb{P}[\sup_{t\in\mathbb{N}_0}\|x(t)\| > \frac{\epsilon}{c_2}] < \bar{p},$$

whenever $\|x_0\| < \min(\delta_1, \delta_2(\frac{\epsilon}{c_2},\bar{p}))$. Now, since $\|x_0\| \leq \frac{\|x_0\|_2}{c_1}$, we have that for all $\epsilon > 0$, $\bar{p} > 0$, the inequality (4) holds whenever $\|x_0\|_2 < \delta(\epsilon,\bar{p}) \triangleq c_1\min(\delta_1, \delta_2(\frac{\epsilon}{c_2},\bar{p}))$, which implies almost sure stability.

Next, we show (5) to establish almost sure *asymptotic* stability of the zero solution. In this regard, first notice that $\mathbb{P}[\lim_{k\to\infty}\eta(k) = 0] = 1$. By using (21), we obtain $\mathbb{P}[\lim_{k\to\infty}\|x(kh)\| = 0] = 1$, which implies $\mathbb{P}[\lim_{t\to\infty}\|x(t)\| = 0] = 1$. Now as a consequence of (33), we have (5). Hence the zero solution of the switched system (1) is asymptotically stable almost surely. □

Theorem 2.4 provides an almost sure asymptotic stability condition for the switched system (1). This result indicates that the stability can be assessed by checking the inequality (17) for all scalars $\rho_q \in [0,1], q \in \mathcal{M}^h$, such that (19), (20) hold. In (17), the scalar $\gamma_q \in \mathbb{R}$ represents the effect of mode sequence q. Specifically, for mode sequences with $\|\Gamma_q\| < 1$, we have $\gamma_q < 0$. A negative value for γ_q implies that the norm of the system's state gets smaller after h time steps, if the mode within those h time steps follows the sequence q. Note that for the case $\Gamma_q = 0$, $\varepsilon \in (0,1)$ in (18) ensures that γ_q is well defined and negative. In practice, ε can be selected as a very small positive number. On the other hand, for mode sequences with $\|\Gamma_q\| > 1$, we have $\gamma_q > 0$. A positive value for γ_q indicates that the mode sequence q may cause the norm of the system's state to increase. Hence, the term $\sum_{q\in\mathcal{M}^h}\gamma_q\rho_q$ in (17) with $\rho_q = \lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\mathbb{1}[\bar{r}(i) = q]$ would correspond to the average of the effects of all h-length sequences in $\mathcal{M}^h$. However, this average cannot be computed directly, since in this paper, we consider the case where the specific values of $\lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\mathbb{1}[\bar{r}(i) = q]$ are not available.

On the other hand, we show by using Lemma 2.3 that if the long-run average activity of modes is known to be bounded as in (2) and (3), then $\rho_q = \lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\mathbb{1}[\bar{r}(i) = q]$ would satisfy (19), (20). Hence, for stability analysis, one can check the sign of $\sum_{q\in\mathcal{M}^h}\gamma_q\rho_q$ in (17) for all $\rho_q \in [0,1], q \in \mathcal{M}^h$, that satisfy (19), (20). This is equivalent to checking the stability for all possible mode sequence scenarios that satisfy (2) and (3), since different values of the limits $\rho_q = \lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$, represent different scenarios. We will show in Section III that rather than checking the condition in (17) for all possible scenarios, we can utilize linear programming methods to identify the worst scenario in terms of stability, and check the condition only for that scenario.

Note that in Theorem 2.4 we require the existence of $\lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\mathbb{1}[\bar{r}(i) = q]$ for all $q \in \mathcal{M}^h$, even though the particular values of these limits are not needed for stability analysis. The following result identifies a class of mode signals $\{r(t) \in \{0,1\}\}_{t\in\mathbb{N}_0}$ for which these limits exist. Using this result, we will show that our analysis technique is applicable in a variety of scenarios.

Proposition 2.5: Let $\{g(t) \in \mathcal{S}\}_{t\in\mathbb{N}_0}$ with $g(0) = g_0 \in \mathcal{S}$ be a finite-state irreducible Markov chain. Assume $\{r(t) \in \mathcal{M}\}_{t\in\mathbb{N}_0}$ is given by

$$r(t) \triangleq \begin{cases} 1, & g(t) \in \mathcal{S}_1,\\ \vdots & \\ M, & g(t) \in \mathcal{S}_M,\end{cases} \quad t \in \mathbb{N}_0, \tag{34}$$

where $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_M$ form a partition of the set $\mathcal{S}$, i.e., $\cup_{i=1}^M\mathcal{S}_i = \mathcal{S}$ and $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$, $i \neq j$. Then for all $h \in \mathbb{N}$, the limits $\lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\mathbb{1}[\bar{l}(i) = q]$, $q \in \mathcal{M}^h$, exist.

The proof of this result is composed of several key steps. The first step is based on the observation that $\bar{r}(\cdot)$ is generated from sequences of values that the process $r(\cdot)$ takes between every h time steps. By exploiting this observation, we construct a new process $\bar{g}(\cdot)$ representing the sequences of values that $g(\cdot)$ takes between every d steps, where d is a carefully chosen period length that is an integer multiple of h. In the second step, we establish the relation between the processes $\bar{r}(\cdot)$ and $\bar{g}(\cdot)$ by using (34). Then, in the final step, we show that $\lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\mathbb{1}[\bar{r}(i) = q]$ can be obtained by utilizing invariant distributions of $\bar{g}(\cdot)$.

Proof: In the proof, we use the notion of *period* for Markov chains [53]. Specifically, the period $\tau_\sigma \in \mathbb{N}$ of a state $\sigma \in \mathcal{S}$ is defined by

$$\tau_\sigma \triangleq \gcd\{t \in \mathbb{N}\colon \mathbb{P}[g(t) = \sigma | g(0) = \sigma] > 0\}, \quad \sigma \in \mathcal{S},$$

where $\gcd(T)$ denotes the greatest common denominator of the elements of the set T. By this definition, the random time intervals between revisits to state σ are guaranteed to be integer multiples of τ_σ. Since $\{g(t) \in \mathcal{S}\}_{t\in\mathbb{N}_0}$ is an irreducible finite-state Markov chain, it follows from Corollary 8.3.7 of [53] that the period is the same for all states. We use $\tau \in \mathbb{N}$ to denote this period, i.e., $\tau = \tau_1 = \tau_2 = \cdots = \tau_{|\mathcal{S}|}$, where $|\mathcal{S}|$ denotes the number of elements in the set $\mathcal{S}$. Now let $d \triangleq \tau h$.

Next, we define a sequence-valued process to characterize the evolution of $g(\cdot)$ in every d steps. To this end, first, for each $\sigma \in \mathcal{S}$, let $\mathcal{I}_{\sigma,k} \triangleq \{s \in \mathcal{S}\colon \mathbb{P}[g(kd) = s | g(0) = \sigma] > 0\}$,

$k \in \{1, \ldots, |\mathcal{S}|\}$, and $\mathcal{I}_\sigma \triangleq \cup_{k=1}^{|\mathcal{S}|} \mathcal{I}_{\sigma,k}$. The set $\mathcal{I}_\sigma \subset \mathcal{S}$ denotes the states that can be reached from the state σ in steps that are integer multiples of d. In addition, for each $\sigma \in \mathcal{S}$, let

$$\bar{\mathcal{S}}_\sigma \triangleq \{(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_d) \colon \bar{s}_j \in \mathcal{S}, j \in \{1, \ldots, d\}, \bar{s}_1 \in \mathcal{I}_\sigma, \\ \mathbb{P}[g(1) = \bar{s}_2, \ldots, g(d-1) = \bar{s}_d | g(0) = \bar{s}_1] > 0\}.$$

Now we define the sequence-valued process $\{\bar{g}(i)\}_{i \in \mathbb{N}_0}$ by

$$\bar{g}(i) \triangleq (g(id), g(id+1), \ldots, g((i+1)d-1)). \tag{35}$$

Notice that $\bar{g}(i) \in \bar{\mathcal{S}}_{g_0}$, $i \in \mathbb{N}_0$.

Our next goal is to show that the sequence-valued Markov chain $\{\bar{g}(i) \in \bar{\mathcal{S}}_{g_0}\}_{i \in \mathbb{N}_0}$ is an irreducible Markov chain. Specifically, we prove that for every $\bar{\sigma}, \bar{s} \in \bar{\mathcal{S}}_{g_0}$, there exists $\bar{k} \in \mathbb{N}$ such that

$$\mathbb{P}[\bar{g}(i + \bar{k}) = \bar{s} \,|\, \bar{g}(i) = \bar{\sigma}] > 0. \tag{36}$$

To this end, first note that $\bar{g}_1(i) \in \mathcal{I}_{g_0}$, $i \in \mathbb{N}_0$, that is, the first elements of the sequence-values that $\bar{g}(\cdot)$ takes are elements of the set $\mathcal{I}_{g_0}$. It follows from the definition of $\mathcal{I}_{g_0}$ that for all $\sigma, s \in \mathcal{I}_{g_0}$,

$$\{k \in \mathbb{N} \colon \mathbb{P}[g(kd) = s | g(0) = \sigma] > 0\} \neq \emptyset.$$

Now, define $k \colon \mathcal{I}_{g_0} \times \mathcal{I}_{g_0} \to \mathbb{N}$ by

$$k(\sigma, s) \triangleq \min\{k \in \mathbb{N} \colon \mathbb{P}[g(kd) = s | g(0) = \sigma] > 0\}. \tag{37}$$

Moreover, note that for any given $\bar{\sigma}, \bar{s} \in \bar{\mathcal{S}}_{g_0}$, we can always pick a state $c \in \mathcal{I}_{g_0}$ and let $\bar{k} \triangleq k(c, \bar{s}_1) + 1$ so that

$$\mathbb{P}[g(1) = c | g(0) = \bar{\sigma}_d] > 0.$$

By (37), we have $\mathbb{P}[g(k(c, \bar{s}_1)d) = \bar{s}_1 | g(0) = c] > 0$, and consequently

$$\begin{aligned}
&\mathbb{P}[\bar{g}(i + \bar{k}) = \bar{s} | \bar{g}(i) = \bar{\sigma}] \\
&\quad = \mathbb{P}[\bar{g}(i + \bar{k}) = \bar{s} | g((i+1)d - 1) = \bar{\sigma}_d] \\
&\quad = \mathbb{P}[g((i + \bar{k})d + 1) = \bar{s}_2, \ldots, \\
&\qquad\quad g((i + \bar{k} + 1)d - 1) = \bar{s}_d | g((i + \bar{k})d) = \bar{s}_1] \\
&\qquad \cdot \mathbb{P}[g((i + \bar{k})d) = \bar{s}_1 | g((i+1)d - 1) = \bar{\sigma}_d] \\
&\quad \geq \mathbb{P}[g((i + \bar{k})d + 1) = \bar{s}_2, \ldots, \\
&\qquad\quad g((i + \bar{k} + 1)d - 1) = \bar{s}_d | g((i + \bar{k})d) = \bar{s}_1] \\
&\qquad \cdot \mathbb{P}[g((i + \bar{k})d) = \bar{s}_1 | g((i+1)d) = c] \\
&\qquad \cdot \mathbb{P}[g((i+1)d) = c | g((i+1)d - 1) = \bar{\sigma}_d] > 0.
\end{aligned}$$

Thus, the sequence-valued Markov chain $\{\bar{g}(i) \in \bar{\mathcal{S}}_{g_0}\}_{i \in \mathbb{N}_0}$ is irreducible. Now, define the function $\alpha : \bar{\mathcal{S}}_{g_0} \times \mathcal{M}^h \to \mathbb{N}_0$ by

$$\alpha(\bar{s}, q) \triangleq \sum_{j=0}^{\tau-1} \mathbb{1}[\bar{s}_{jh+1} \in \mathcal{S}_{q_1}, \ldots, \bar{s}_{jh+h} \in \mathcal{S}_{q_h}], \tag{38}$$

for $\bar{s} \in \bar{\mathcal{S}}_{g_0}, q \in \mathcal{M}^h$. Note that $\alpha(\bar{s}, q) \in \mathbb{N}_0$ is the number of times the sequence q appears in the process $r(\cdot)$, when the process $g(\cdot)$ takes the values $\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_d$. This number is computed by dividing the d-length sequence $\bar{s}$ into τ number of h-length sequences and counting the number of h-length sequences whose elements are from sets $\mathcal{S}_{q_1}, \mathcal{S}_{q_2}, \ldots, \mathcal{S}_{q_h}$.

By using (38), we get

$$\frac{1}{\tau k} \sum_{i=0}^{\tau k - 1} \mathbb{1}[\bar{r}(i) = q] = \frac{1}{\tau k} \sum_{i=0}^{k-1} \sum_{\bar{s} \in \bar{\mathcal{S}}_{g_0}} \mathbb{1}[\bar{g}(i) = \bar{s}] \alpha(\bar{s}, q). \tag{39}$$

Let $\bar{\pi}_{g_0, \bar{s}} \in [0, 1]$, $\bar{s} \in \bar{\mathcal{S}}_{g_0}$, denote the invariant distribution associated with $\{\bar{g}(i) \in \bar{\mathcal{S}}_{g_0}\}_{i \in \mathbb{N}_0}$. It then follows from ergodic theorem for finite-state Markov chains (see Theorem 1.10.2 of [50]) that

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \sum_{\bar{s} \in \bar{\mathcal{S}}_{g_0}} \mathbb{1}[\bar{g}(i) = \bar{s}] \alpha(\bar{s}, q) = \sum_{\bar{s} \in \bar{\mathcal{S}}_{g_0}} \bar{\pi}_{g_0, \bar{s}} \alpha(\bar{s}, q).$$

Hence, as a consequence of (39), $\lim_{k \to \infty} \frac{1}{\tau k} \sum_{i=0}^{\tau k - 1} \mathbb{1}[\bar{r}(i) = q]$ exists and is given by

$$\lim_{k \to \infty} \frac{1}{\tau k} \sum_{i=0}^{\tau k - 1} \mathbb{1}[\bar{r}(i) = q] = \frac{1}{\tau} \sum_{\bar{s} \in \bar{\mathcal{S}}_{g_0}} \bar{\pi}_{g_0, \bar{s}} \alpha(\bar{s}, q). \tag{40}$$

Our final goal is to show

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] = \lim_{k \to \infty} \frac{1}{\tau k} \sum_{i=0}^{\tau k - 1} \mathbb{1}[\bar{r}(i) = q].$$

To this end, first let $\theta(k) \triangleq \lfloor \frac{k}{\tau} \rfloor$ and observe that

$$\begin{aligned}
\frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] &= \frac{1}{k} \sum_{i=0}^{\tau\theta(k) - 1} \mathbb{1}[\bar{r}(i) = q] \\
&\quad + \frac{1}{k} \sum_{i=\tau\theta(k)}^{k-1} \mathbb{1}[\bar{r}(i) = q], \quad k \in \mathbb{N}_0.
\end{aligned} \tag{41}$$

Since $\mathbb{1}[\bar{r}(i) = q] \in \{0, 1\}$, we have $0 \leq \frac{1}{k} \sum_{i=\tau\theta(k)}^{k-1} \mathbb{1}[\bar{r}(i) = q] \leq \frac{\tau}{k}$, and hence, $0 \leq \lim_{k \to \infty} \frac{1}{k} \sum_{i=\tau\theta(k)}^{k-1} \mathbb{1}[\bar{r}(i) = q] \leq \lim_{k \to \infty} \frac{\tau}{k} = 0$. As a result,

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=\tau\theta(k)}^{k-1} \mathbb{1}[\bar{r}(i) = q] = 0. \tag{42}$$

Furthermore, we have

$$\begin{aligned}
&\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{\tau\theta(k) - 1} \mathbb{1}[\bar{r}(i) = q] \\
&\quad = \lim_{k \to \infty} \frac{1}{k} \frac{\tau\theta(k)}{\tau\theta(k)} \sum_{i=0}^{\tau\theta(k) - 1} \mathbb{1}[\bar{r}(i) = q] \\
&\quad = \lim_{k \to \infty} \frac{\tau\theta(k)}{k} \lim_{k \to \infty} \frac{1}{\tau\theta(k)} \sum_{i=0}^{\tau\theta(k) - 1} \mathbb{1}[\bar{r}(i) = q].
\end{aligned} \tag{43}$$

Since $\lim_{k \to \infty} \frac{\tau\theta(k)}{k} = 1$, by (40) and (43),

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{\tau\theta(k) - 1} \mathbb{1}[\bar{r}(i) = q] = \frac{1}{\tau} \sum_{\bar{s} \in \bar{\mathcal{S}}_{g_0}} \bar{\pi}_{g_0, \bar{s}} \alpha(\bar{s}, q). \tag{44}$$

It then follows from (41), (42), and (44) that $\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ exists and is given by

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] = \frac{1}{\tau} \sum_{\bar{s} \in \bar{\mathcal{S}}_{g_0}} \bar{\pi}_{g_0, \bar{s}} \alpha(\bar{s}, q).$$

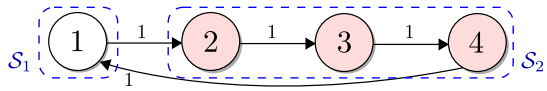

Figure 1. Transition diagram for Markov chain $\{g(t) \in \mathcal{S} = \{1,2,3,4\}\}_{t\in\mathbb{N}_0}$ with initial condition $g(0) = 1$ and partitions $\mathcal{S}_1 = \{1\}$, $\mathcal{S}_2 = \{2,3,4\}$, characterizing a 4-periodic mode sequence $1,2,2,2,1,2,2,2,\ldots$.

□

Remark 2.6: In Proposition 2.5, we provide a characterization of the mode signal $\{r(t) \in \{1,\ldots,M\}\}_{t\in\mathbb{N}_0}$ through an irreducible Markov chain $\{g(t) \in \mathcal{S}\}_{t\in\mathbb{N}_0}$. The set $\mathcal{S}$ of the possible values of $g(\cdot)$ is the union of disjoint sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_M$. By the definition in (34), the mode signal takes the value s, when $g(t) \in \mathcal{S}_s$. Observe that this mode signal follows the hidden Markov model [54], and even though $\{g(t) \in \mathcal{S}\}_{t\in\mathbb{N}_0}$ is a Markov chain, $\{r(t) \in \mathcal{M}\}_{t\in\mathbb{N}_0}$ may fail to be so as we will show in two examples shortly below. To see this, we first express the switched system (1) equivalently by the alternative system

$$x(t+1) = \bar{A}_{g(t)}x(t), \tag{45}$$

with $\bar{A}_j \triangleq A_i$, $j \in \mathcal{S}_i$. Here, (45) is a switched system where the mode signal $\{g(t) \in \mathcal{S}\}_{t\in\mathbb{N}_0}$ is a Markov chain. This switched system is also called a Markov jump system [13]. We would like to highlight that the stability results on Markov jump systems are not applicable here. Different from the ordinary Markov jump systems, neither transition probabilities nor stationary distributions of the process $g(\cdot)$ are available for analysis. In fact even the size of the set $\mathcal{S}$ may not be known. In particular, when we consider the application to networked control under jamming attacks (see Sections IV and V), the process $\{g(t) \in \mathcal{S}\}_{t\in\mathbb{N}_0}$ may characterize a jamming attacker's strategy and its properties are not available for analysis.

Furthermore, we remark that (34) is only one of the possible characterizations of the mode signal for our analysis to be applicable. The mode signal may also be generated in other ways following different random or deterministic characterizations. Our stability analysis in Theorem 2.4 relies on the bounds on the average number of times each mode is active in the long run (see Assumption 2.1) instead of the properties of particular mode signal characterizations.

The characterization through (34) is general enough to model various types of mode signals. We present two examples in this regard.

Example 1: Periodic mode switchings can be described with an irreducible and periodic Markov chain $\{g(t) \in \mathcal{S}\}_{t\in\mathbb{N}_0}$. Consider for example a switched system with 2 modes. The mode sequence is assumed to repeat itself in every 4 time steps. Specifically, in every 4 time steps, mode 1 is active for 1 time step then mode 2 becomes active for the next 3 time steps. This periodic switching scenario can be characterized by setting $\mathcal{S}_1 \triangleq \{1\}$, $\mathcal{S}_2 \triangleq \{2,3,4\}$, and $\{g(t) \in \mathcal{S}\}_{t\in\mathbb{N}_0}$ as a Markov chain with transition probabilities shown on the edges of the transition graph in Fig. 1. In this situation $g(\cdot)$ repeatedly takes the values $1,2,3,4,1,2,3,4,\ldots$. As a result, by the definition in (34), the mode signal $r(\cdot)$ takes the values $1,2,2,2,1,2,2,2,\ldots$ indicating the periodic change in the mode.

In this example, the switched system can be used for modeling a networked control system under periodic attacks. In particular, the first mode corresponds to a successful packet exchange between the plant and the controller, and the second mode represents the dynamics when there is a transmission failure due to attacks. The characterization of the mode signal through the setup in Fig. 1 represents an example of the discrete-time version of the periodic attacks discussed in [19]. Here the attacker periodically repeats sleeping for 1 time step and emitting a jamming signal to block network transmissions for 3 consecutive time steps. It is important to note that when the networked control system is periodically attacked, the specific failure sequence and the period itself are part of attacker's strategy and in general they are not available to the system operator. We consider a networked control problem that covers this case in Section IV. There, we show that to check networked control system's stability through Theorem 2.4, only the knowledge of the upper-bound on the average attack ratio is needed. For the periodic attack in Fig. 1, the upper-bound on this ratio is given by $\overline{\rho}_2$, since the second mode corresponds to the attacks. Notice that in this case we can select $\underline{\rho}_1 = 1 - \overline{\rho}_2$, $\overline{\rho}_1 = 1$, and $\underline{\rho}_2 = 0$.

Example 2: The characterization in (34) can also be used to describe random packet transmission failures. For example, communication channels following the Markov model can be described simply by setting $\mathcal{S}_1 \triangleq \{1\}$, $\mathcal{S}_2 \triangleq \{2\}$, and $\{g(t) \in \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2\}_{t\in\mathbb{N}_0}$ as a Markov chain with certain transition probabilities. In addition, the Gilbert-Elliott model and other more advanced models based on Markov chains (see [45], [46]) can also be described within the framework. For instance, in Gilbert-Elliott model, the channel is in the state of either "Good" or "Bad". In Good channel state, packet losses occur with a small probability e; moreover, in Bad channel state, failure probability denoted by f may be large. Transitions between Good and Bad states occur with probability p from Good to Bad and q from Bad to Good. This scenario can be described by setting $\mathcal{S}_1 \triangleq \{1,2\}$, $\mathcal{S}_2 \triangleq \{3,4\}$, and $\{g(t) \in \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2\}_{t\in\mathbb{N}_0}$ as a Markov chain with transition diagram shown in Fig. 2. In this setting, $g(t) \in \{1,3\}$ corresponds to Good channel state and $g(t) \in \{2,4\}$ corresponds to Bad. On the other hand, by (34), $g(t) \in \mathcal{S}_2$ indicates a packet exchange failure at time t, whereas $g(t) \in \mathcal{S}_1$ indicates a successful packet exchange attempt. Using different settings for $\mathcal{S}_1$, $\mathcal{S}_2$, and $\{g(t) \in \mathcal{S}\}_{t\in\mathbb{N}_0}$, we can also model the situation where the network faces both jamming attacks and random packet transmission failures.

Note that when $\{r(t) \in \mathcal{M}\}_{t\in\mathbb{N}_0}$ is characterized through (34), the limits $\lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$, exist for all $h \in \mathbb{N}$. Hence, in such cases, the stability analysis in Theorem 2.4 can be conducted with any $h \in \mathbb{N}$. On the other hand, for other characterizations of $\{r(t) \in \mathcal{M}\}_{t\in\mathbb{N}_0}$, it may be the case that the limits exist for $h \in \{1,2,\ldots,\hat{h}\}$ but not for $h > \hat{h}$, where $\hat{h} \in \mathbb{N}$. In those situations, Theorem 2.4 is applicable only for $h \in \{1,2,\ldots,\hat{h}\}$.

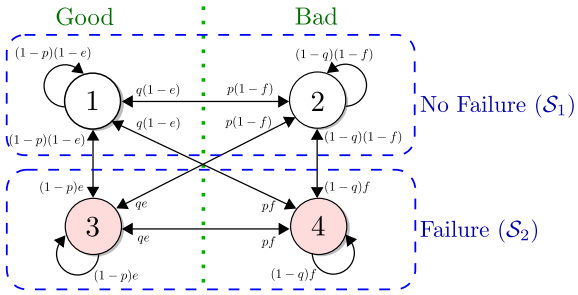

Figure 2. Transition diagram for Markov chain $\{g(t) \in \mathcal{S} = \{1,2,3,4\}\}_{t\in\mathbb{N}_0}$ with partitions $\mathcal{S}_1 = \{1,2\}, \mathcal{S}_2 = \{3,4\}$, characterizing packet exchange failures on a Gilbert-Elliott channel.

III. Linear Programming Methods for Stability Assessment

In this section, we investigate two closely-related linear programming problems and present a method for checking the almost sure asymptotic stability condition given in Theorem 2.4 through their optimal solutions.

A. Linear Programming Problem 1

Theorem 2.4 states that the switched system (1) is stable if there exists an induced matrix norm $\|\cdot\|$ and a scalar $\varepsilon \in (0,1)$ such that the inequality (17) holds for all $\rho_q \in [0,1], q \in \mathcal{M}^h$, that satisfy (19), (20). In what follows, we provide a linear programming problem to check this condition for a given induced matrix norm $\|\cdot\|$ and scalar $\varepsilon \in (0,1)$.

Now define $\gamma_q \in \mathbb{R}$, $q \in \mathcal{M}^h$, as in (18) and consider the linear programming problem

$$\begin{array}{ll} \underset{\rho_q\in[0,1],\, q\in\mathcal{M}^h}{\text{maximize}} & \sum_{q\in\mathcal{M}^h} \gamma_q \rho_q \\ \text{subject to} & (19), (20). \end{array} \tag{46}$$

For the stability analysis, different values of $\rho_q, q \in \mathcal{M}^h$, that satisfy (19), (20) represent possible mode activity scenarios such that the long run average conditions (2) and (3) hold. The linear programming problem (46) allows us to identify scenarios that maximize $\sum_{q\in\mathcal{M}^h} \gamma_q \rho_q$. We can then check the stability condition (17) with the maximum value of $\sum_{q\in\mathcal{M}^h} \gamma_q \rho_q$ instead of checking it for all possible scenarios.

In the following lemma, we show that the linear programming problem (46) is feasible, that is, there always exist $\rho_q \in [0,1], q \in \mathcal{M}^h$, that satisfy (19), (20). Furthermore, we show that the problem is bounded (i.e., the objective function $\sum_{q\in\mathcal{M}^h} \gamma_q \rho_q$ in (46) is bounded).

Lemma 3.1: The linear programming problem (46) is feasible and bounded.

Proof: First, we show that the feasible region of the linear programming problem is not empty. To this end, first observe that $\sum_{s=1}^{M} \underline{\rho}_s \le 1 \le \sum_{s=1}^{M} \overline{\rho}_s$. This is because Assumption 2.1 implies

$$\begin{aligned} 1 &= \limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \sum_{s=1}^{M} \mathbb{1}[r(t)=s] \\ &\le \sum_{s=1}^{M} \limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[r(t)=s] \le \sum_{s=1}^{M} \overline{\rho}_s, \end{aligned}$$

and similarly,

$$\begin{aligned} 1 &= \liminf_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \sum_{s=1}^{M} \mathbb{1}[r(t)=s] \\ &\ge \sum_{s=1}^{M} \limsup_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[r(t)=s] \ge \sum_{s=1}^{M} \underline{\rho}_s. \end{aligned}$$

Now let $\underline{\varrho} \triangleq \sum_{s=1}^{M} \underline{\rho}_s$, $\overline{\varrho} \triangleq \sum_{s=1}^{M} \overline{\rho}_s$ and $\beta_s \triangleq (\overline{\rho}_s - \underline{\rho}_s)\frac{1-\underline{\varrho}}{\overline{\varrho}-\underline{\varrho}}$ and consider $\rho_q, q \in \mathcal{M}^h$, given by

$$\rho_q = \begin{cases} \underline{\rho}_s + \beta_s, & \text{if } c_s(q) = h,\ s \in \mathcal{M}, \\ 0, & \text{otherwise.} \end{cases} \tag{47}$$

To establish that the feasible region contains $\rho_q, q \in \mathcal{M}^h$, given by (47), we show that (19) and (20) hold. First, (19) holds because

$$\begin{aligned} \sum_{q\in\mathcal{M}^h} \rho_q &= \sum_{s=1}^{M} (\underline{\rho}_s + \beta_s) = \underline{\varrho} + \frac{1-\underline{\varrho}}{\overline{\varrho}-\underline{\varrho}} \sum_{s=1}^{M} (\overline{\rho}_s - \underline{\rho}_s) \\ &= \underline{\varrho}\left(1 - \frac{1-\underline{\varrho}}{\overline{\varrho}-\underline{\varrho}}\right) + \frac{1-\underline{\varrho}}{\overline{\varrho}-\underline{\varrho}}\overline{\varrho} = 1. \end{aligned} \tag{48}$$

Now we show (20). Since $\underline{\varrho} \le 1 \le \overline{\varrho}$, we have $\beta_s \in [0, \overline{\rho}_s - \underline{\rho}_s]$. It then follows that

$$\sum_{q\in\mathcal{M}^h} c_s(q)\rho_q = \underline{\rho}_s + \beta_s \in [\underline{\rho}_s, \overline{\rho}_s], \quad s \in \mathcal{M}. \tag{49}$$

It now remains to show that the solutions to the linear programming problem are bounded. Note that since $\rho_q \le 1$, $q \in \mathcal{M}^h$, we have $\sum_{q\in\mathcal{M}^h} \gamma_q \rho_q \le (\max_{q\in\mathcal{M}^h} \gamma_q) \sum_{q\in\mathcal{M}^h} \rho_q \le (\max_{q\in\mathcal{M}^h} \gamma_q) M^h < \infty$, which completes the proof. □

Lemma 3.1 implies that there exists an optimal solution to the linear programming problem (46) (see Proposition 3.1 of [55]). Even though there may be multiple optimal solutions, we can always compute the optimal value of the objective function using any one of those solutions. Let J_h denote the optimal value of the objective function when h-length sequences are considered, that is,

$$J_h \triangleq \max\Big\{ \sum_{q\in\mathcal{M}^h} \gamma_q \rho_q \colon \rho_q \in [0,1], q \in \mathcal{M}^h, \text{ s.t. } (19), (20) \Big\}.$$

The stability of the switched system (1) can be assessed by checking the sign of the optimal value J_h. Specifically, the zero solution $x(t) \equiv 0$ of the switched system (1) is asymptotically stable almost surely if

$$J_h < 0. \tag{50}$$

This is because (50) implies that (17) in Theorem 2.4 holds for all $\rho_q \in [0,1], q \in \mathcal{M}^h$, that satisfy (19), (20).

Remark 3.2: The optimal solution J_h for the linear programming problem (46) may be positive when h is small, but may become negative for sufficiently large h. The reason is that with large h, stability/instability properties of more mode activity patterns are taken into account. For instance, consider a switched system with two modes. When $h = 2$, the effects of the dynamics associated with packet failure sequences in $\mathcal{M}^2 \triangleq \{(1,1),(1,2),(2,1),(2,2)\}$ are represented by $\gamma_q, q \in$

$\mathcal{M}^2$. However, $\gamma_q, q \in \mathcal{M}^2$, cannot be used to distinguish the difference between stabilizing (destabilizing) effects of longer mode activity sequences $(1,2,2,1)$ and $(2,1,1,2)$, since both of them are composed of the same smaller sequences $(1,2)$, $(2,1)$. Thus to show stability, we may need to take into account longer mode sequences and obtain J_h for larger values of $h \in \mathbb{N}$. Note that γ_q associated with a mode sequence $q = (1,2,2,1)$ may be negative, even though $\gamma_{(1,2)}$ and $\gamma_{(2,1)}$ associated with the smaller sequences $(1,2)$, $(2,1)$ are positive. This is similar to the observation that a switched system with individually unstable modes may be stable if the switching is constrained in a certain way (see Chapter 2 of [5]). We note that our approach of choosing a larger h value is related to the approach utilized in [39] for showing average contractivity of Markov jump systems over h steps as well as the approaches used in [40]–[42] for showing switched system stability through the investigation of the evolution of the mode signal over multiple time steps. In addition to these works, a related approach was also used in [43], [44] to show convergence of risk-sensitive and risk sensitive like filters by means of establishing strict contractivity of Riccati recursions over h steps.

Even though there are efficient algorithms for solving linear programming problems, it is difficult to solve (46) and obtain J_h when $h \in \mathbb{N}$ is large. This is because the number of variables $\rho_q, q \in \mathcal{M}^h$, of the problem (46) grows exponentially in h. Specifically, the number of elements of the set $\mathcal{M}^h$, and hence the number of variables of the linear programming problem (46) is given by $f(h,M) \triangleq M^h$. In the following, we show that an alternative linear programming problem with fewer variables shares the same optimal objective function value as that of (46). In particular, the number of variables in this alternative problem grows only polynomially in h.

B. Linear Programming Problem 2

We observe in the linear programming problem (46) that due to the particular structure of our problem setting, some of the variables have the same coefficients in the constraints. In particular, if two (or more) mode sequences are reordered versions of each other, then the variables associated with those mode sequences have the same coefficients in the constraints. This allows us to obtain an alternative problem with fewer variables.

For an intuitive explanation of how this is possible, consider the case with $M = 2$, $h = 2$. In this case, the constraints in (20) are $\underline{\rho}_1 \leq \rho_{(1,1)} + 0.5\rho_{(1,2)} + 0.5\rho_{(2,1)} + 0\rho_{(2,2)} \leq \overline{\rho}_1$ and $\underline{\rho}_2 \leq 0\rho_{(1,1)} + 0.5\rho_{(1,2)} + 0.5\rho_{(2,1)} + 1\rho_{(2,2)} \leq \overline{\rho}_2$, where the variables $\rho_{(1,2)}$ and $\rho_{(2,1)}$ have the same coefficients. If, for example, $\gamma_{(2,1)} \geq \gamma_{(1,2)}$, then it means that we can maximize the objective $\gamma_{(1,1)}\rho_{(1,1)} + \gamma_{(1,2)}\rho_{(1,2)} + \gamma_{(2,1)}\rho_{(2,1)} + \gamma_{(2,2)}\rho_{(2,2)}$ by setting $\rho_{(1,2)}$ to 0 and choosing $\rho_{(2,1)}$ as large as we can within the constraints. This is because if $\rho_q = \tilde{\rho}_q$, $q \in \mathcal{M}^h$, is an optimal solution of the problem, then $\rho_q = \hat{\rho}_q, q \in \mathcal{M}^h$, with $\hat{\rho}_{(1,1)} = \tilde{\rho}_{(1,1)}$, $\hat{\rho}_{(1,2)} = 0$, $\hat{\rho}_{(2,1)} = \tilde{\rho}_{(1,2)} + \tilde{\rho}_{(2,1)}$, and $\hat{\rho}_{(2,2)} = \tilde{\rho}_{(2,2)}$ is also an optimal solution since $\rho_q = \hat{\rho}_q, q \in \mathcal{M}^h$, satisfy the constraints and $\gamma_{(2,1)} \geq \gamma_{(1,2)}$ implies $\sum_{q\in\mathcal{M}^h} \gamma_q\tilde{\rho}_q \leq \sum_{q\in\mathcal{M}^h} \gamma_q\hat{\rho}_q$. (Furthermore, if $\gamma_{(2,1)}$ is strictly larger than $\gamma_{(1,2)}$, then for all optimal solutions ρ_q, $q \in \mathcal{M}^h$, the value of $\rho_{(1,2)}$ is necessarily 0.) As this discussion indicates, the variable $\rho_{(1,2)}$ can be removed from the optimization problem for this example, since the terms that involve $\rho_{(1,2)}$ are all 0. Similarly, if $\gamma_{(2,1)} \leq \gamma_{(1,2)}$, then the variable $\rho_{(2,1)}$ can be removed. Notice that similar techniques can be used for all $M \in \mathbb{N}$ and $h \in \mathbb{N}$. Furthermore, more variable reductions are achieved as M and h increase. In particular, we obtain the following linear programming problem:

Let $\mathcal{Z}_h \triangleq \{(z_1, z_2, \ldots, z_M) \colon z_s \in \{0,1,\ldots,h\}, s \in \mathcal{M}, \sum_{s=1}^{M} z_s = h\}$, and consider the linear programming problem

$$\begin{array}{ll} \underset{\rho'_z\in[0,1],\, z\in\mathcal{Z}_h}{\text{maximize}} & \sum_{z\in\mathcal{Z}_h} \gamma'_z\rho'_z \\ \text{subject to} & \sum_{z\in\mathcal{Z}_h} \rho'_z = 1, \qquad (a) \\ & \underline{\rho}_s \leq \sum_{z\in\mathcal{Z}_h} \frac{z_s}{h}\rho'_z \leq \overline{\rho}_s,\ s \in \mathcal{M},\ (b) \end{array} \tag{51}$$

where

$$\gamma'_z \triangleq \max_{q\in\mathcal{M}^{h,z}} \gamma_q, \quad z \in \mathcal{Z}_h, \tag{52}$$

with $\mathcal{M}^{h,z} \triangleq \{q \in \mathcal{M}^h : c_1(q) = z_1, c_2(q) = z_2, c_3(q) = z_3 \ldots, c_M(q) = z_M\}$, $z \in \mathcal{Z}_h$.

In what follows, we first show that the objective functions of the linear programming problems (46) and (51) have the same optimal values. After that we discuss the advantage of the linear programming problem (51) over the problem (46). Specifically, we show that it is easier to solve the linear programming problem (51) because it involves fewer variables.

Lemma 3.3: The linear programming problem (51) is feasible and bounded.

Proof: The proof is similar to that of Lemma 3.1. First, we show that the feasible region of the linear programming problem (51) is not empty. To this end, consider ρ'_z, $z \in \mathcal{Z}_h$, given by

$$\rho'_z = \sum_{q\in\mathcal{M}^{h,z}} \rho_q, \tag{53}$$

with ρ_q given in (47). Notice that $\rho'_z \in [0,1]$, $z \in \mathcal{Z}_h$. This is because $\rho_q \geq 0$, $q \in \mathcal{M}^h$, implies $\rho'_z \geq 0$, and moreover, $\rho'_z = \sum_{q\in\mathcal{M}^{h,z}} \rho_q \leq \sum_{q\in\mathcal{M}^h} \rho_q = 1$ due to (48). To establish that the feasible region contains $\rho'_z, z \in \mathcal{Z}_h$, given by (53), we show that (51a) and (51b) hold. Now, (51a) holds, because it follows from (48) that

$$\sum_{z\in\mathcal{Z}_h} \rho'_z = \sum_{z\in\mathcal{Z}_h} \sum_{q\in\mathcal{M}^{h,z}} \rho_q = \sum_{q\in\mathcal{M}^h} \rho_q = 1,$$

where we also used the fact that $\mathcal{M}^{h,z} \cap \mathcal{M}^{h,\hat{z}} = \emptyset$ for $z \neq \hat{z}$, $z, \hat{z} \in \mathcal{Z}_h$, and $\cup_{z\in\mathcal{Z}_h} \mathcal{M}^{h,z} = \mathcal{M}^h$.

Next, to show (51b), note that $\sum_{z\in\mathcal{Z}_h} \frac{z_s}{h}\rho'_z = \sum_{z\in\mathcal{Z}_h} \sum_{q\in\mathcal{M}^{h,z}} \frac{z_s}{h}\rho_q = \sum_{q\in\mathcal{M}^h} \frac{c_s(q)}{h}\rho_q$. Hence, (49) implies (51b).

It now remains to show that the solutions to the linear programming problem are bounded. Note that since $\rho'_z \leq 1$, $z \in \mathcal{Z}_h$, it follows from (52) that $\sum_{z\in\mathcal{Z}_h} \gamma'_z\rho'_z \leq (\max_{z\in\mathcal{Z}_h} \gamma'_z) \sum_{z\in\mathcal{Z}_h} \rho'_z \leq (\max_{q\in\mathcal{M}^h} \gamma_q)\binom{h+M-1}{M-1} < \infty$, which completes the proof. $\square$

By Lemma 3.3, there exists an optimal solution to the linear programming problem (51). Let J'_h denote the optimal value of the objective function for a given $h \in \mathbb{N}$, that is,

$$J'_h \triangleq \max\Big\{ \sum_{z\in\mathcal{Z}_h} \gamma'_z \rho'_z \colon \rho'_z, z \in \mathcal{Z}_h, \text{ s.t. } (51a), (51b) \Big\}.$$

Lemma 3.4: The objective functions of the linear programming problems (46) and (51) have the same optimal values, that is, $J_h = J'_h$.

Proof: We prove this result by showing $J_h \leq J'_h$ and $J_h \geq J'_h$ separately.

To establish $J_h \leq J'_h$, we show that for all $\rho_q \in [0,1], q \in \mathcal{M}^h$, such that (19), (20) hold, we have $\sum_{q\in\mathcal{M}^h} \gamma_q \rho_q \leq J'_h$. Now, notice that for all $\rho_q \in [0,1], q \in \mathcal{M}^h$, such that (19), (20) hold, the constraints in (51a) and (51b) hold with $\rho'_z \triangleq \sum_{q\in\mathcal{M}^{h,z}} \rho_q$, $z \in \mathcal{Z}_h$, and as a result, $\sum_{z\in\mathcal{Z}_h} \gamma'_z \rho'_z \leq J'_h$. Hence, for all $\rho_q, q \in \mathcal{M}^h$, such that (19), (20) hold, we have

$$\begin{aligned}\sum_{q\in\mathcal{M}^h} \gamma_q \rho_q &= \sum_{z\in\mathcal{Z}_h} \sum_{q\in\mathcal{M}^{h,z}} \gamma_q \rho_q \leq \sum_{z\in\mathcal{Z}_h} \sum_{q\in\mathcal{M}^{h,z}} \gamma'_z \rho_q \\ &= \sum_{z\in\mathcal{Z}_h} \gamma'_z \sum_{q\in\mathcal{M}^{h,z}} \rho_q = \sum_{z\in\mathcal{Z}_h} \gamma'_z \rho'_z \leq J'_h,\end{aligned}$$

which implies $J_h \leq J'_h$.

To prove $J_h \geq J'_h$, we now show that there exists $\rho_q \in [0,1], q \in \mathcal{M}^h$, such that $\sum_{q\in\mathcal{M}^h} \gamma_q \rho_q = J'_h$ and (19), (20) hold. Here, let $\hat{\rho}'_z, z \in \mathcal{Z}_h$, denote an optimal solution to the linear programming problem (51), that is,

$$\sum_{i\in\mathcal{Z}_h} \gamma'_z \hat{\rho}'_z = J'_h.$$

Now, for each $z \in \mathcal{Z}_h$, let $q^{(z)} \in \operatorname{argmax}_{q\in\mathcal{M}^{h,z}} \gamma_q$ and set $\rho_{q^{(z)}} = \hat{\rho}'_z$, $\rho_q = 0$, $q \in \mathcal{M}^{h,z} \setminus \{q^{(z)}\}$. It follows that $\rho_q, q \in \mathcal{M}^h$, satisfy (19), (20); furthermore,

$$\sum_{q\in\mathcal{M}^h} \gamma_q \rho_q = \sum_{z\in\mathcal{Z}_h} \gamma'_z \hat{\rho}'_z = J'_h.$$

This establishes that there exist $\rho_q \in [0,1], q \in \mathcal{M}^h$, such that $\sum_{q\in\mathcal{M}^h} \gamma_q \rho_q = J'_h$ and (19), (20) hold, which implies that $J_h \geq J'_h$. □

A direct consequence of Lemma 3.4 is that if $J'_h < 0$, then the zero solution $x(t) \equiv 0$ of the switched system (1) is asymptotically stable almost surely.

So far we established that almost sure asymptotic stability of the switched system (1) can be assessed by checking signs of the optimal objective function values (J_h and J'_h) of linear programming problems (46) and (51). The switched system (1) is stable if the value of $J_h = J'_h$ is negative.

We observe that solving the linear programming problem (51) can be computationally more advantageous in comparison to the problem (46). This is because (51) involves fewer variables. Specifically, the number of elements of the set $\mathcal{Z}_h$, and hence the number of variables of the linear programming problem (51), is $f'(h,M) \triangleq \binom{h+M-1}{M-1} = \frac{(h+M-1)!}{(M-1)!h!}$. For h and M larger than 1, the number of variables in the problem (51) is strictly smaller than that of the problem (46), that is,

$$f'(h,M) < f(h,M), \quad h > 1, \quad M > 1.$$

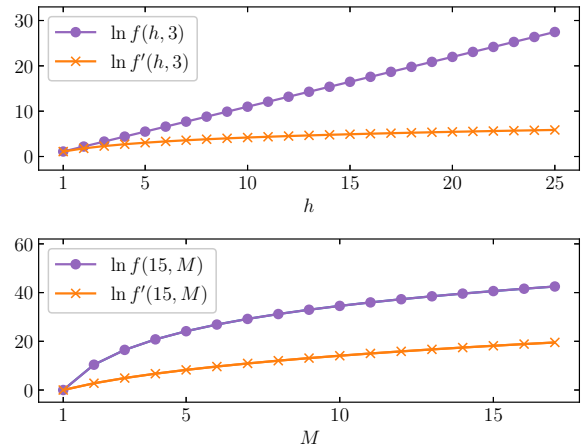

Figure 3. Comparison of $\ln f(h,M)$ and $\ln f'(h,M)$ (Top: varying h, fixed $M = 3$; Bottom: fixed $h = 15$, varying M).

We note again that f grows *exponentially* in h. On the other hand f', the number of variables in the problem (51), grows only *polynomially* in h. Specifically, we have

$$f'(\alpha h, M) \leq \alpha^{M-1} f'(h,M), \quad \alpha, h, M \in \mathbb{N}.$$

As a result, when h is large, obtaining J'_h is much faster than obtaining J_h. We also remark that the number of variables grows polynomially in the number of modes M as well. In particular, we have

$$f'(h, \alpha M) \leq \alpha^h f'(h,M), \quad \alpha, h, M \in \mathbb{N}.$$

Fig. 3 shows graphs of $\ln f(h,M)$ and $\ln f'(h,M)$ indicating how the numbers of variables in problems (46) and (51) grow with respect to h and M. The difference becomes larger as M and h get larger. For instance, in the case where $h = 15$ and $M = 3$, there are $f(15,3) = 14{,}348{,}907$ variables in the problem (46), whereas (51) involves only $f'(15,3) = 136$ variables.

Notice that an optimal solution $\hat{\rho}'_z, z \in \mathcal{Z}_h$, to the linear programming problem (51) can be used to obtain an optimal solution $\hat{\rho}_q, q \in \mathcal{M}^h$, to the problem (46) by selecting $q^{(z)} \in \operatorname{argmax}_{q\in\mathcal{M}^{h,z}} \gamma_q$ for each $z \in \mathcal{Z}_h$ and then setting $\hat{\rho}_q, q \in \mathcal{M}^h$, as $\hat{\rho}_{q^{(z)}} = \hat{\rho}'_z$, $\hat{\rho}_q = 0$, $q \in \mathcal{M}^{h,z} \setminus \{q^{(z)}\}$, $z \in \mathcal{Z}_h$. As established in the second part of the proof of Lemma 3.4, $\hat{\rho}_q, q \in \mathcal{M}^h$, satisfy $\sum_{q\in\mathcal{M}^h} \gamma_q \hat{\rho}_q = J'_h$, and hence provide an optimal solution to the problem (46), since $J_h = J'_h$. Clearly, if for any $z \in \mathcal{Z}_h$ the set $\operatorname{argmax}_{q\in\mathcal{M}^{h,z}} \gamma_q$ has more than one element, then there are multiple optimal solutions to the problem (46).

It is important to note that although solving the linear programming problem (51) is easier due to fewer variables, it requires precomputation of coefficients γ'_z by (52). Notice that, for each z, the complexity of computing γ'_z is linear in the number of variables of the set $\mathcal{M}^{h,z}$, which is given by $\frac{h!}{z_1! z_2! \cdots z_M!}$. It turns out that this computation can be carried out for all coefficients γ'_z at the same time using parallel computing techniques. For instance, in the case of $h = 15$ and $M = 3$, the set $\mathcal{Z}_h$ of $f'(15,3) = 136$ variables can be partitioned into 8 subsets with size 17. We can then utilize a computer with 8 central processing units to carry

out the computation for each of these subsets. Both linear programming problems (46) and (51) require calculation of norms of matrix products in the computation of γ_q, $q \in \mathcal{M}^h$, given in (18). For large values of h, this is a computationally intensive calculation, but it can also be conducted in parallel for different q values.

We also note that using different matrix norms in the definition of $\gamma_q, q \in \mathcal{M}^h$, can be useful to check stability in the case of limited computational resources. This is because J_h and J'_h may be positive for a particular matrix norm and negative for another. Note also that $\gamma_q, q \in \mathcal{M}^h$, depend on the value of ε in (18) if $\Gamma_q = 0$ for some $q \in \mathcal{M}^h$. If $\Gamma_q = 0$ for some $q \in Q_h$ and the optimal solution value $J_h = J'_h$ is positive for a particular value of ε, then we can try solving the linear programming problems with a smaller ε value. If the optimal solution value $J_h = J'_h$ is negative for a value $\varepsilon = \hat{\varepsilon}$, then the stabilization is guaranteed. In that case, we do not need to consider smaller ε values, since for every $\varepsilon \in (0, \hat{\varepsilon})$, the optimal solution value $J_h = J'_h$ is also guaranteed to be negative. This is because the constraints in the linear programming problems do not change with ε and the coefficients of the objective functions for the case with $\varepsilon \in (0, \hat{\varepsilon})$ are smaller than the case with $\varepsilon = \hat{\varepsilon}$.

We remark that for stability analysis, the sign of J_h and J'_h can also be assessed by solving linear feasibility problems without computing the actual values of those scalars. In particular, we have $J_h \geq 0$ if and only if there exist $\rho_q \in [0,1]$, $q \in \mathcal{M}^h$, such that (19), (20), and $\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q \geq 0$ hold. Similarly, we have $J'_h \geq 0$ if and only if there exist $\rho'_z \in [0,1]$, $z \in \mathcal{Z}_h$, such that (51a), (51b), and $\sum_{z \in \mathcal{Z}_h} \gamma'_z \rho'_z \geq 0$ hold. Note that solving these feasibility problems is not necessarily faster in comparison to solving the associated linear programming problems, since the numbers of variables in these two feasibility problems are equal to those in the associated linear programming problems.

IV. Application to Networked Control Under Jamming Attacks

In this section we consider two problem settings where we model the networked control system as a switched system.

A. Control over Delay-Free Communication Links

First, we explore the networked control problem where at each time instant, the plant and the controller attempt to exchange state and control input packets over a communication channel. In this problem setting, network transmissions do not face delay, but packet exchange attempts between the plant and the controller may be subject to packet losses due to malicious jamming attacks or nonmalicious communication errors. In a successful packet exchange attempt, the plant transmits the state information to the controller; the controller uses the received state information to compute the control input through a linear control law and sends back the control input to the plant. The transmitted control input is then applied at the plant side. Packet exchange attempt failures happen when either the measured state packets or the control input packets are lost. In that case, the control input at the plant side is set to 0.

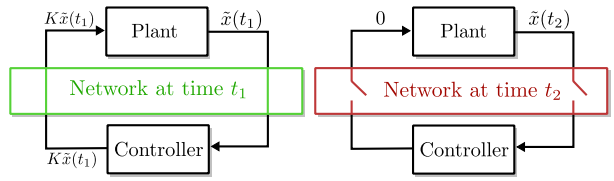

Figure 4. Packet exchange success mode (left) and failure mode (right) of the networked control system.

Fig. 4 illustrates the operation of the networked control system during a successful packet exchange attempt at time t_1 and a failed exchange attempt at time t_2.

The dynamics of the linear plant is given by

$$\tilde{x}(t+1) = A\tilde{x}(t) + Bu(t), \quad \tilde{x}(0) = \tilde{x}_0, \quad t \in \mathbb{N}_0, \tag{54}$$

where $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$ and $u(t) \in \mathbb{R}^m$ denote the state and the control input, respectively; furthermore, $A \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ and $B^{\tilde{n} \times m}$ are the state and input matrices, respectively.

We use the binary-valued process $\{l(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$ to describe success or failure states of packet exchange attempts. Specifically, the state $l(t) = 0$ indicates that the packet exchange attempt at time t is successful, whereas $l(t) = 1$ indicates failure. In this case, the control input $u(t)$ applied at the plant side is given by

$$u(t) \triangleq (1 - l(t))\, K\tilde{x}(t), \quad t \in \mathbb{N}_0, \tag{55}$$

where $K \in \mathbb{R}^{m \times \tilde{n}}$ denotes the feedback gain.

In [21], we proposed a characterization for $\{l(t) \in \{0,1\}\}_{t \in \mathbb{N}_0}$ that allows us to model the effects of random packet losses and jamming attacks in a unified manner. This characterization relies on the following assumption.

Assumption 4.1: There exists a scalar $\rho \in [0,1]$ such that

$$\sum_{k=1}^{\infty} \mathbb{P}[\sum_{i=0}^{k-1} l(i) > \rho k] < \infty. \tag{56}$$

Here, the inequality (56) can be considered as a condition on the evolution of tail probability $\mathbb{P}\big[\sum_{i=0}^{k-1} l(i)/k > \rho\big]$. The scalar $\rho \in [0,1]$ in (56) plays a key role in characterizing a probabilistic bound on the average ratio of packet exchange failures. Observe that the case where all packet transmission attempts result in failure can be described by setting $\rho = 1$. It is shown in [21] that ρ in (56) can be obtained to be strictly smaller than 1 for certain random and malicious packet loss models. These models include time-inhomogeneous Markov chains for describing random packet losses, as well as a discrete-time version of the malicious attack model in [20], where the number of packet exchange attempts that face attacks is upper bounded by a certain ratio of the total number of packet exchange attempts.

We showed in [21] that the closed-loop networked control system is stable, when ρ in (56) takes a sufficiently small value. In what follows we provide an alternative stability analysis method for the networked control system by utilizing Theorem 2.4. This new method turns out to be less conservative than the results in [21] in certain scenarios. To utilize Theorem 2.4 we first describe the closed-loop system as a discrete-time switched system.

The networked control system (54), (55) is equivalently described as a discrete-time switched system (1) with the state $x(t) = \tilde{x}(t)$ and the mode signal given by $r(t) = l(t)+1$. Furthermore, the subsystem matrices are given by $A_1 = A+BK$, $A_2 = A$. Now, under Assumption 4.1, the inequalities (2) and (3) in Assumption 2.1 hold with $\underline{\rho}_1 = 1-\rho, \overline{\rho}_1 = 1, \underline{\rho}_2 = 0$, and $\overline{\rho}_2 = \rho$. This is because (56) implies

$$\limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} l(t) \le \rho, \tag{57}$$

and hence $\limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \mathbb{1}[r(t) = 2] \le \rho$. Through this switched system characterization, stability of the networked control system (54), (55) can be analyzed by using Theorem 2.4. Moreover, the linear programming problems developed in Section III can also be employed.

In [21] (see also [27]), an event-triggering controller is used for stabilization, and the packet exchange attempt times are decided by utilizing a set of triggering conditions. These conditions can be adjusted to consider the problem setting where the plant and the controller attempt packet exchanges at each time instant. For this problem setting, Theorem 2.4 is less conservative and can be considered as an enhancement of the stability result presented in [21]. In fact, the stability result in [21] is obtained by analyzing the evolution of a Lyapunov-like function $V(x) = x^{\mathrm{T}}Px$ at each time step. This analysis idea can be recovered by our approach presented in this paper through setting $h = 1$, and defining the norm in (18) as the matrix norm induced by the vector norm $\|x\|_P \triangleq \sqrt{x^{\mathrm{T}}Px}$. In particular, for the case where $\lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} l(i)$ exists, for all scenarios in which the stability condition in Theorem 3.5 in [21] holds, the stability condition in Theorem 2.4 also holds. Furthermore, as we illustrate in Section V, there are cases where the condition in Theorem 3.5 in [21] does not hold but the condition in Theorem 2.4 is satisfied.

The following result shows that when $\lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} l(i)$ exists, for all scenarios in which the stability condition in Theorem 3.5 in [21] holds, the stability condition in Theorem 2.4 also holds.

Proposition 4.1: Assume $\lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} l(i)$ exists. Suppose the stability condition in Theorem 3.5 in [21] holds, that is, there exist a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ and scalars $\beta \in (0,1)$, $\varphi \in [1,\infty)$ such that

$$\beta P - (A+BK)^{\mathrm{T}} P (A+BK) \ge 0, \tag{58}$$

$$\varphi P - A^{\mathrm{T}} P A \ge 0, \tag{59}$$

$$(1-\rho)\ln\beta + \rho\ln\varphi < 0, \tag{60}$$

hold. Then with $h \triangleq 1$ the stability condition in Theorem 2.4 also holds, i.e., $\lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$, exist; moreover, there is a matrix norm $\|\cdot\|$ and a scalar $\varepsilon \in (0,1)$ such that (17) holds for all $\rho_q \in [0,1]$, $q \in \mathcal{M}^h$, that satisfy (19) and (20).

Proof: First, note that when $h = 1$, we have $\mathcal{M}^h = \{(1),(2)\}$. As a result, existence of $\lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} l(i)$ implies that $\lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$, also exist.

Next, let the norm in (18) be the matrix norm induced by the vector norm $\|x\|_P \triangleq \sqrt{x^{\mathrm{T}}Px}$, that is, $\|M\| \triangleq \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\|Mx\|_P}{\|x\|_P}$, $M \in \mathbb{R}^{n\times n}$. Under this norm, the inequalities (58) and (59) imply $\|A_1\| = \|A+BK\| \le \sqrt{\beta}$ and $\|A_2\| = \|A\| \le \sqrt{\varphi}$. Now let $\varepsilon \triangleq \sqrt{\beta}$. Since $\varepsilon = \sqrt{\beta} < \sqrt{\varphi}$, by using (60) and $\rho_{(2)} \le \overline{\rho}_2 = \rho$, we get

$$\begin{aligned}
&\sum_{q\in\mathcal{M}^h} \gamma_q\rho_q = \gamma_{(1)}\rho_{(1)} + \gamma_{(2)}\rho_{(2)} = \gamma_{(1)}(1-\rho_{(2)}) + \gamma_{(2)}\rho_{(2)} \\
&= (1-\rho_{(2)})\ln\max\{\|A+BK\|,\varepsilon\} + \rho_{(2)}\ln\max\{\|A\|,\varepsilon\} \\
&\le (1-\rho_{(2)})\ln\sqrt{\beta} + \rho_{(2)}\ln\sqrt{\varphi} \\
&\le (1-\rho)\ln\sqrt{\beta} + \rho\ln\sqrt{\varphi} = \frac{1}{2}((1-\rho)\ln\beta + \rho\ln\varphi) < 0,
\end{aligned}$$

for all $\rho_q \in [0,1]$, $q \in \mathcal{M}^h = \{(1),(2)\}$, such that (19) and (20) hold, which completes the proof. □

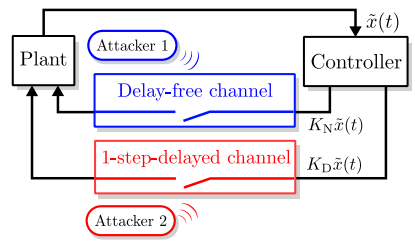

Figure 5. Networked control over delay-free and 1-step-delayed channels.

B. Control over Delay-Free and One-Step Delayed Communication Links

The switched system framework generalizes [21] to the multiple mode case with $M > 2$. This aspect is now illustrated through the networked control system depicted in Fig. 5, where the control actions are transmitted to the plant over two separate communication channels. We assume that one of these channels faces no delay and the other one faces 1-step delay in transmissions. Investigation of a networked control setup involving multiple channels with different delays is useful for analyzing systems that incorporate multiple actuators placed at different locations. The nodes that relay the information coming from the controller to certain actuators may induce delays due to different security measures in transmission powers, encryptions, and so on.

In our problem setting, the plant is as given in (54). The controller receives the system state $\tilde{x}(t)$ at each time t, and computes two control inputs $K_{\mathrm{N}}\tilde{x}(t)$ and $K_{\mathrm{D}}\tilde{x}(t)$ that are attempted to be transmitted on the delay-free and 1-step-delayed channels, respectively. We respectively use $\{l_{\mathrm{N}}(t) \in \{0,1\}\}_{t\in\mathbb{N}_0}$ and $\{l_{\mathrm{D}}(t) \in \{0,1\}\}_{t\in\mathbb{N}_0}$ to indicate failures on the delay-free and the one-step-delayed channels. If both channels fail ($l_{\mathrm{N}}(t) = 1$, $l_{\mathrm{D}}(t) = 1$), the control input at the plant side is set to 0. Furthermore, the control data $K_{\mathrm{D}}\tilde{x}(t-1)$ received from the delayed channel is used only if the transmission on the delay-free channel fails ($l_{\mathrm{N}}(t) = 1$, $l_{\mathrm{D}}(t) = 0$). Otherwise (when $l_{\mathrm{N}}(t) = 0$, $l_{\mathrm{D}}(t) = 1$ or $l_{\mathrm{N}}(t) = 0$, $l_{\mathrm{D}}(t) = 0$), the control input at the plant side is set to $K_{\mathrm{N}}\tilde{x}(t)$ received from the delay-free channel. Hence, the control input applied at the plant is given by

$$u(t) = (1-l_{\mathrm{N}}(t))K_{\mathrm{N}}\tilde{x}(t) + l_{\mathrm{N}}(t)(1-l_{\mathrm{D}}(t))K_{\mathrm{D}}\tilde{x}(t-1), \tag{61}$$

for $t \geq 1$. Assuming $u(0) = 0$, the closed-loop dynamics (54), (61) can be given by

$$\begin{aligned}&\begin{bmatrix} \tilde{x}(t+2) \\ \tilde{x}(t+1) \end{bmatrix} \\ &= \begin{bmatrix} A + (1 - l_{\mathrm{N}}(t))BK_{\mathrm{N}} & l_{\mathrm{N}}(t)(1 - l_{\mathrm{D}}(t))BK_{\mathrm{D}} \\ I_{\tilde{n}} & 0 \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} \tilde{x}(t+1) \\ \tilde{x}(t) \end{bmatrix}, \quad t \in \mathbb{N}_0. \end{aligned} \tag{62}$$

By setting

$$x(t) \triangleq \begin{bmatrix} \tilde{x}(t+1) \\ \tilde{x}(t) \end{bmatrix}, \; r(t) \triangleq \begin{cases} 1, & l_{\mathrm{N}}(t) = 0, \\ 2, & l_{\mathrm{N}}(t) = 1, \, l_{\mathrm{D}}(t) = 0, \\ 3, & l_{\mathrm{N}}(t) = 1, \, l_{\mathrm{D}}(t) = 1, \end{cases} \tag{63}$$

for $t \in \mathbb{N}_0$, the closed-loop dynamics (62) forms a switched system (1) with 3 modes represented by

$$\begin{aligned} A_1 &\triangleq \begin{bmatrix} A + BK_{\mathrm{N}} & 0 \\ I_{\tilde{n}} & 0 \end{bmatrix}, \\ A_2 &\triangleq \begin{bmatrix} A & BK_{\mathrm{D}} \\ I_{\tilde{n}} & 0 \end{bmatrix}, \quad A_3 \triangleq \begin{bmatrix} A & 0 \\ I_{\tilde{n}} & 0 \end{bmatrix}. \end{aligned} \tag{64}$$

Concerning the communication channels in the networked control system, we assume the following.

Assumption 4.2: There exist scalars $\sigma_{\mathrm{N}}, \sigma_{\mathrm{D}}, \rho_{\mathrm{N}}, \rho_{\mathrm{D}} \in [0, 1]$ such that

$$\liminf_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l_{\mathrm{N}}(t) \geq \sigma_{\mathrm{N}}, \; \limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l_{\mathrm{N}}(t) \leq \rho_{\mathrm{N}}, \tag{65}$$

$$\liminf_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l_{\mathrm{D}}(t) \geq \sigma_{\mathrm{D}}, \; \limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l_{\mathrm{D}}(t) \leq \rho_{\mathrm{D}}, \tag{66}$$

hold almost surely.

Note that this assumption provides lower- and upper-bounds on the long-run average numbers of transmission failures on the delay-free and 1-step-delayed channels.

In the following result, we show that under Assumption 4.2, the mode signal of the switched system representing the networked control system satisfies Assumption 2.1.

Proposition 4.2: Suppose (65) and (66) in Assumption 4.2 are satisfied. Then (2) and (3) in Assumption 2.1 hold with

$$\underline{\rho}_1 = 1 - \rho_{\mathrm{N}}, \, \overline{\rho}_1 = 1 - \sigma_{\mathrm{N}}, \tag{67}$$

$$\underline{\rho}_2 = \max\{0, \sigma_{\mathrm{N}} - \rho_{\mathrm{D}}\}, \, \overline{\rho}_2 = \min\{\rho_{\mathrm{N}}, 1 - \sigma_{\mathrm{D}}\}, \tag{68}$$

$$\underline{\rho}_3 = \max\{0, \sigma_{\mathrm{N}} + \sigma_{\mathrm{D}} - 1\}, \, \overline{\rho}_3 = \min\{\rho_{\mathrm{N}}, \rho_{\mathrm{D}}\}. \tag{69}$$

Proof: We use Lemma A.1 to show the result. To this end, note that

$$\mathbb{1}[r(t) = 1] = 1 - l_{\mathrm{N}}(t), \tag{70}$$

$$\mathbb{1}[r(t) = 2] = l_{\mathrm{N}}(t)(1 - l_{\mathrm{D}}(t)), \tag{71}$$

$$\mathbb{1}[r(t) = 3] = l_{\mathrm{N}}(t) l_{\mathrm{D}}(t), \quad t \in \mathbb{N}_0. \tag{72}$$

First, we show (2) and (3) hold for the case $s = 1$ with $\underline{\rho}_1, \overline{\rho}_1$ given in (67). By (65) and (66), the inequalities (74) and (75) in Lemma A.1 hold with $\xi_1(\cdot), \xi_2(\cdot)$ given by $\xi_1(t) = 1 - l_{\mathrm{N}}(t)$, $\xi_2(t) = 1$, $t \in \mathbb{N}_0$, and $\varsigma_1 = 1 - \rho_{\mathrm{N}}$, $\varrho_1 = 1 - \sigma_{\mathrm{N}}$, $\varsigma_2 = 1$, $\varrho_2 = 1$. By the lemma, we obtain (2) from (76) and (3) from (77) with $\underline{\rho}_1, \overline{\rho}_1$ given in (67).

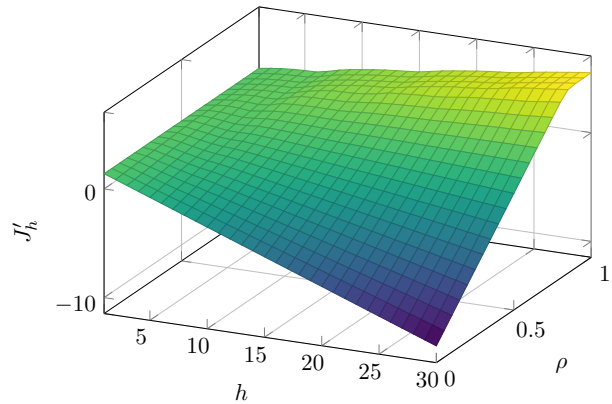

Figure 6. Optimal solution value J'_h of the linear programming problem (51) with respect to h and ρ.

Next, we show (2) and (3) hold for the case $s = 2$ with $\underline{\rho}_2, \overline{\rho}_2$ given in (68). By (65) and (66), the inequalities (74) and (75) hold with $\xi_1(\cdot), \xi_2(\cdot)$ given by $\xi_1(t) = l_{\mathrm{N}}(t)$, $\xi_2(t) = 1 - l_{\mathrm{D}}(t)$, $t \in \mathbb{N}_0$, and $\varsigma_1 = \sigma_{\mathrm{N}}$, $\varrho_1 = \rho_{\mathrm{N}}$, $\varsigma_2 = 1 - \rho_{\mathrm{D}}$, $\varrho_2 = 1 - \sigma_{\mathrm{D}}$. By applying Lemma A.1, we obtain (2) from (76) and (3) from (77) with $\underline{\rho}_2, \overline{\rho}_2$ given in (68). The result for the other case ($s = 3$) is obtained similarly by using Lemma A.1 together with (72). □

Proposition 4.2 shows that the networked control system (54), (61) with communication channels satisfying Assumption 4.2 can be represented by a switched system with a mode signal that satisfies Assumption 2.1. As a result, Theorem 2.4 and the linear programming problems developed in Section III can be used for the stability analysis.

V. Numerical Examples

In this section, we illustrate the efficacy of our results by investigating stability properties of networked control systems discussed in Sections IV-A and IV-B.

A) Example 1: Consider the system (54) with

$$A = \begin{bmatrix} 1 & 0.1 \\ -0.5 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 1.2 \end{bmatrix}. \tag{73}$$

In [21], we explored stabilization of this system over a network that faces random and malicious packet losses. There, we proposed a linear state feedback controller with feedback gain $K = \begin{bmatrix} -2.9012 & -0.9411 \end{bmatrix}$. By utilizing Theorem 3.5 of [21], we see that the closed-loop system is almost surely asymptotically stable whenever the scalar ρ identified in Assumption 4.1 is inside the range $[0, 0.411]$. In the following we show that even for strictly larger values of ρ, the closed-loop system remains almost surely asymptotically stable.

For investigating stability of the closed-loop system (54), (55), we first characterize it as a switched system (1) with two modes represented with $A_1 = A + BK$ and $A_2 = A$. Notice that for this switched system, Assumption 4.1 implies (57), and as a result, Assumption 2.1 holds with $\underline{\rho}_1 = 1 - \rho$, $\overline{\rho}_1 = 1$, $\underline{\rho}_2 = 0$, and $\overline{\rho}_2 = \rho$.

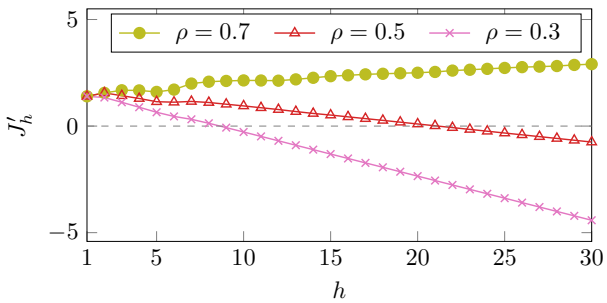

Figure 7. Optimal solution value J'_h of the linear programming problem (51) with respect to h for $\rho = 0.3$, $\rho = 0.5$ and $\rho = 0.7$.

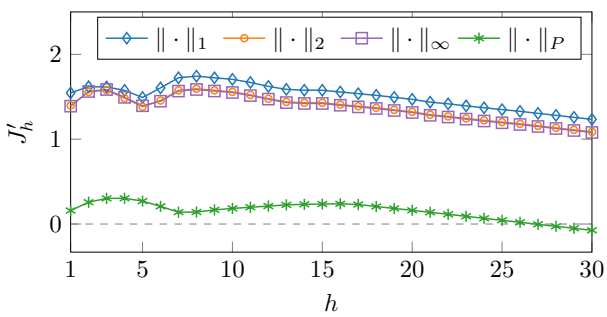

Figure 8. Optimal solution value J'_h of the linear programming problem (51) with respect to h for fixed $\rho = 0.6$ and different norms used in (18).

We numerically solve the linear programming problems (46) and (51) to obtain J_h and J'_h for different values of ρ and h. For finding the coefficients γ_q and γ'_z of the objective functions, we use (18) (with the matrix norm induced by the Euclidean norm and $\varepsilon = 10^{-24}$) and (52). We numerically confirm that $J_h = J'_h$ for $h = \{1, \ldots, 11\}$. For $h \geq 12$, we utilize only the linear programming problem (51) and obtain J'_h, as solving the problem (46) takes excessively long times. We see in Fig. 6 that for smaller values of ρ, J'_h takes negative values indicating almost sure asymptotic stability. In particular, we see in Fig. 7 that when $\rho = 0.5$, we obtain $J'_{22} < 0$, which implies that (17) holds for all $\rho_q \in [0, 1], q \in \mathcal{M}^{22}$, that satisfy (19), (20). It follows from Theorem 2.4 that if Assumption 4.1 holds with $\rho = 0.5$, and $\lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[\bar{l}(t) = q]$ exists for each $q \in \mathcal{M}^{22}$, then the zero solution of the closed-loop system is almost surely asymptotically stable. We note again that for $\rho = 0.5$, the stability conditions of Theorem 3.5 in [21] do not hold. This indicates that the stability conditions obtained in this paper are less conservative than those in [21].

Note that for a given ρ, obtaining a nonnegative value for $J_h = J'_h$ does not necessarily imply that the system is unstable. For the same ρ, the value of J'_h may be positive for small h and negative for sufficiently large h (see Remark 3.2). For instance, in this example, J'_{10} takes a negative value for $\rho = 0.3$, but not for $\rho = 0.5$ or $\rho = 0.7$ (Fig. 7). If one can only compute J'_h up to $h = 10$ due to limited computational power, then stability for $\rho = 0.5$ cannot be concluded.

We also remark that using different matrix norms in the definition of $\gamma_q, q \in \mathcal{M}^h$, given in (18) results in different trajectories for J'_h. For illustration, we compute J'_h with matrix norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$ (induced respectively by the 1-norm, the Euclidean norm, and the infinity norm of vectors), as well as the matrix norm $\|\cdot\|_P$ induced by the vector norm $\|x\|_P \triangleq \sqrt{x^{\mathrm{T}} P x}$. Here, we use the positive-definite matrix P that we previously utilized in [21] for the Lyapunov-based stability analysis of this system. Fig. 8 shows the optimal solution value J'_h of the linear programming problem (51) for different values of h when $\rho = 0.6$. For this example, the values of J'_h obtained with the matrix norm $\|\cdot\|_P$ is clearly lower than others. We also observe that J'_{30} obtained with $\|\cdot\|_P$ is negative. Thus, we can conclude that (17) holds for all $\rho_q \in [0, 1], q \in \mathcal{M}^{30}$, that satisfy (19), (20). It then follows from Theorem 2.4 if Assumption 4.1 holds with $\rho = 0.6$, and $\lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[\bar{l}(t) = q]$ exists for each $q \in \mathcal{M}^{30}$, then the zero solution of the closed-loop system is almost surely asymptotically stable. Hence, for instance, the system is stable under all periodic attack scenarios with $\rho \in [0, 0.6]$, since Proposition 2.5 implies that the limits $\lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[\bar{l}(t) = q]$, $q \in \mathcal{M}^{30}$, exist in the periodic case.

In fact this stability region $[0, 0.6]$ is quite tight, as there exists a *destabilizing* attack strategy for $\rho = 0.64$. This attack strategy is periodic with a period of 150 time steps. It was identified through the solution $\rho_z, z \in \mathcal{Z}_{30}$, to the linear programming problem (51) solved with $h = 30$ by using the matrix norm $\|\cdot\|_2$ and the bounds $\underline{\rho}_1 = 1 - \rho$, $\overline{\rho}_1 = 1$, $\underline{\rho}_2 = 0, \overline{\rho}_2 = \rho$, where $\rho = 0.64$. Under this attack strategy, the mode signal of the associated switched system (1) repeats the same pattern in every 150 time steps. This pattern is composed of a particular sequence $\tilde{q} \in \mathcal{M}^{30}$ appearing once and then another sequence $\hat{q} \in \mathcal{M}^{30}$ appearing 4 times. Note that under this attack strategy, Assumption 4.1 holds with $\rho = 0.64$, and furthermore, the monodromy matrix associated with the closed-loop periodic networked control system possesses an eigenvalue that is outside the unit circle of the complex plane indicating divergence of the state.

B) Example 2: In this example we demonstrate the results for the networked control setup discussed in Section IV-B. Specifically, we consider the plant with A and B given by (73) in the previous subsection. The control packets are transmitted to the plant over the delay-free and the 1-step-delayed channels depicted in Fig. 5. The feedback gains associated with these channels are given by $K_{\mathrm{N}} = \begin{bmatrix} -2.9012 & -0.9411 \end{bmatrix}$ and $K_{\mathrm{D}} = \begin{bmatrix} -0.04 & -0.3 \end{bmatrix}$. We note that K_{N} is the gain from the previous subsection, and K_{D} ensures that A_2 (of the equivalent switched system formulation (1) with (64)) is a Schur matrix.

We consider the case where the channels are subject to coordinated periodic jamming attacks. In this case, $\{l_{\mathrm{N}}(t) \in \{0, 1\}\}_{t\in\mathbb{N}_0}$ and $\{l_{\mathrm{D}}(t) \in \{0, 1\}\}_{t\in\mathbb{N}_0}$, the failure indicators of the delay-free and the one-step-delayed channels, can be given by

$$l_{\mathrm{N}}(t) = \begin{cases} 1, & g(t) \in \overline{\mathcal{S}}_{\mathrm{N}}, \\ 0, & \text{otherwise}, \end{cases} \quad l_{\mathrm{D}}(t) = \begin{cases} 1, & g(t) \in \overline{\mathcal{S}}_{\mathrm{D}}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\{g(t) \in \mathcal{S}\}_{t\in\mathbb{N}_0}$ is a finite-state irreducible and periodic Markov chain with transition probabilities either 0 or 1, and moreover, $\overline{\mathcal{S}}_{\mathrm{N}}$ and $\overline{\mathcal{S}}_{\mathrm{D}}$ are subsets of $\mathcal{S}$. Fig. 9 shows the transition diagram of $\{g(t) \in \mathcal{S}\}_{t\in\mathbb{N}_0}$ with an 8-periodic pattern. At

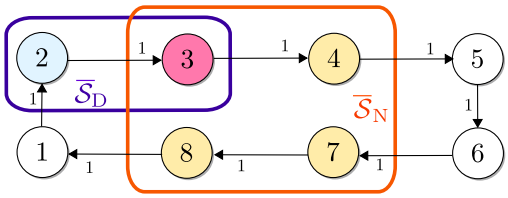

Figure 9. Transition diagram for $\{g(t) \in \mathcal{S}\}_{t\in\mathbb{N}_0}$ representing an 8-periodic jamming attack strategy against the networked control system depicted in Fig 5. With initial condition $g(0) = 1$, the delay-free channel is periodically attacked at times $8t+2$, $8t+3$, $8t+6$, $8t+7$, and moreover, the 1-step-delayed channel is periodically attacked at times $8t+1$, $8t+2$, for $t \in \mathbb{N}_0$.

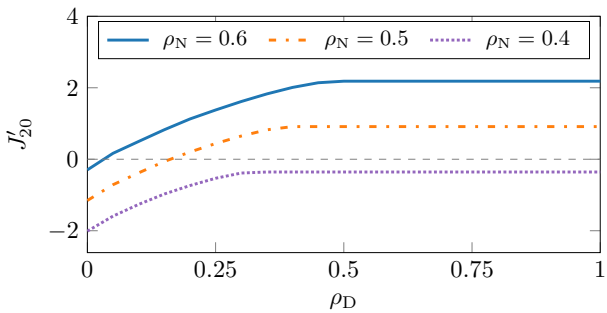

Figure 10. Optimal solution value J'_{20} of the linear programming problem (51) with respect to ρ_{D} for the cases $\rho_{\mathrm{N}} = 0.4$, $\rho_{\mathrm{N}} = 0.5$, and $\rho_{\mathrm{N}} = 0.6$. (We set $\sigma_{\mathrm{D}} = \sigma_{\mathrm{N}} = 0$ for all cases.)

time t, the transmission on the delay-free channel is attacked if $g(t) \in \overline{\mathcal{S}}_{\mathrm{N}} = \{3, 4, 7, 8\}$; moreover, the transmission on the 1-step-delayed channel is attacked if $g(t) \in \overline{\mathcal{S}}_{\mathrm{D}} = \{2, 3\}$, and both channels are attacked when $g(t) = 3$. Note that for any attack strategy represented with an irreducible $\{g(t) \in \mathcal{S}\}_{t\in\mathbb{N}_0}$, the mode signal given by (63) satisfies (34) with $\mathcal{S}_1 \triangleq \mathcal{S} \setminus \overline{\mathcal{S}}_{\mathrm{N}}$, $\mathcal{S}_2 \triangleq \overline{\mathcal{S}}_{\mathrm{N}} \cap (\mathcal{S} \setminus \overline{\mathcal{S}}_{\mathrm{D}})$, and $\mathcal{S}_3 \triangleq \overline{\mathcal{S}}_{\mathrm{N}} \cap \overline{\mathcal{S}}_{\mathrm{D}}$. Hence, it follows from Proposition 2.5 that for all $h \in \mathbb{N}$, the limits $\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$, exist.

To investigate the stability of the networked control system, we consider different scenarios where the long-run-average transmission failures on both channels satisfy Assumption 4.2. Then, Proposition 4.2 implies that the switched system representation satisfies Assumption 2.1 with $\underline{\rho}_s, \overline{\rho}_s$, $s \in \{1, 2, 3\}$, given by (67)–(69) as functions of $\sigma_{\mathrm{N}}, \sigma_{\mathrm{D}}, \rho_{\mathrm{N}}, \rho_{\mathrm{D}}$.

We first consider the case where the information about the average number of failures on delay-free and 1-step-delayed channels is limited. In particular, we set the lower-bounds on the long-run average number of failures on both channels to 0, that is, $\sigma_{\mathrm{N}} = \sigma_{\mathrm{D}} = 0$. Our goal is to identify upper-bounds on the long-run average number of failures of the channels (ρ_{N} and ρ_{D}) for which the closed-loop networked control system is stable. To this end, we solve the linear programming problem (51) for different values of ρ_{N} and ρ_{D}. For finding the coefficients $\gamma'_z = \max_{q\in\mathcal{M}^{h,z}} \gamma_q$, $z \in \mathcal{Z}_h$, of the objective function, we use (18) (with the matrix norm induced by the Euclidean norm and $\varepsilon = 10^{-24}$).

Fig. 10 shows how the optimal objective function J'_{20} of the linear programming problem (51) changes with respect to ρ_{D} for the values $\rho_{\mathrm{N}} = 0.4$, $\rho_{\mathrm{N}} = 0.5$, and $\rho_{\mathrm{N}} = 0.6$. Observe that when the long-run average number of failures on the delay-free

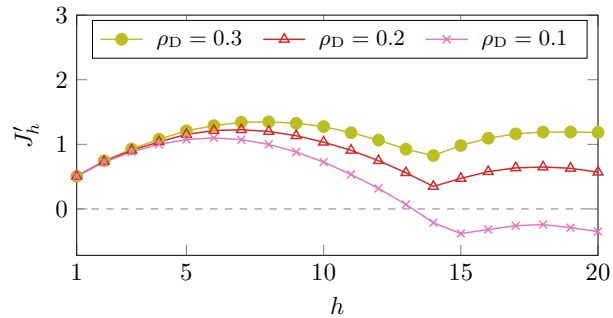

Figure 11. Optimal solution value J'_h of the linear programming problem (51) with respect to h for the cases $\rho_{\mathrm{D}} = 0.1$, $\rho_{\mathrm{D}} = 0.2$, and $\rho_{\mathrm{D}} = 0.3$. (We set $\sigma_{\mathrm{D}} = 0$, $\sigma_{\mathrm{N}} = \rho_{\mathrm{N}} = 1$ for all cases.)

communication channel is sufficiently small, stability can be achieved regardless of the amount of transmission failures on the 1-step-delayed channel. This is seen in Fig. 10 for the case $\rho_{\mathrm{N}} = 0.4$. Specifically, we have $J'_{20} < 0$, which implies that (17) holds for all $\rho_q \in [0, 1], q \in \mathcal{M}^{20}$, that satisfy (19), (20). Hence, the stability can be concluded by Theorem 2.4 for all $\rho_{\mathrm{D}} \in [0, 1]$. On the other hand, when the delay-free channel faces more failures, the transmissions on the 1-step-delayed channel become more important. In particular for $\rho_{\mathrm{N}} = 0.5$ and $\rho_{\mathrm{N}} = 0.6$, the value of J'_{20} is negative and the stability can be concluded *only* for sufficiently small values of ρ_{D}.

Next, we consider the situation where the delay-free channel is known to be completely blocked due to jamming attacks at each time instant. To explore this case, we set $\sigma_{\mathrm{N}} = \rho_{\mathrm{N}} = 1$. In this setup, all control packets are to be transmitted over the 1-step-delayed channel. We would like to find out the long-run average number of failures that can be tolerated on this channel. For this purpose, we solve the linear programming problem (51) to obtain J'_h with respect to h for three different cases: $\rho_{\mathrm{D}} = 0.1$, $\rho_{\mathrm{D}} = 0.2$, and $\rho_{\mathrm{D}} = 0.3$. In all cases we set $\sigma_{\mathrm{D}} = 0$. Fig. 11 shows J'_h for different values of ρ_{D}. We observe that $J'_{14} < 0$ when $\rho_{\mathrm{D}} = 0.1$, implying that (17) holds for all $\rho_q \in [0, 1], q \in \mathcal{M}^{14}$, that satisfy (19), (20). Hence, by Theorem 2.4 the zero solution of the closed-loop system is almost surely asymptotically stable if the ratio of the transmission failures on the 1-step-delayed channel is bounded by 0.1 in the long-run. On the other hand, for $\rho_{\mathrm{D}} = 0.2$ and $\rho_{\mathrm{D}} = 0.3$, we cannot guarantee stability, since we have $J'_h > 0$, $h \in \{1, \ldots, 20\}$.

C) Example 3: Consider the system (54), (55) with

$$A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, K = \begin{bmatrix} -a_1 & -a_2 \end{bmatrix},$$

where $a_1 = 2$, $a_2 = 1$. Suppose that the failure indicator process $\{l(t) \in \{0, 1\}\}_{t\in\mathbb{N}_0}$ satisfies Assumption 4.1, where ρ denotes the probabilistic bound on the average ratio of failures. This networked control system can be written as a switched system (1) with

$$A_1 = A + BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}.$$

Following the formulation in Section IV-A, let $\underline{\rho}_1 \triangleq 1 - \rho, \overline{\rho}_1 \triangleq 1, \underline{\rho}_2 \triangleq 0$, and $\overline{\rho}_2 \triangleq \rho$. It follows from (57) that

Assumption 2.1 is satisfied. Our goal is to check the stability of this system by using Theorem 2.4. To this end we utilize the linear programming problem (46). Notice that for this example, with $h = 2$, we have

$$\Gamma_{(1,1)} = A_1^2 = 0$$

in (7). Hence, $\gamma_{(1,1)}$, in (18) depends on the particular value selected for ε. With the matrix norm induced by the Euclidean norm and $\varepsilon = 10^{-24}$, we solve the linear programming problem (46) and obtain $J_2 > 0$ for every $\rho \geq 0.5$. For instance, we have $J_2 = 0.8047$ for $\rho = 0.5$, $J_2 = 0.9328$ for $\rho = 0.6$, and $J_2 = 1.0609$ for $\rho = 0.7$. Notice that when $\rho \geq 0.5$, either one of the switching sequences $2, 1, 2, 1, \ldots$ or $1, 2, 1, 2, \ldots$ are allowed. These sequences destabilize the system. As a result J_2 cannot be negative when $\rho \geq 0.5$. In particular, when $\rho = 0.5$, an optimal solution is given by $\rho_{(2,1)} = 1.0$ and $\rho_q = 0$ for $q \neq (2, 1)$, which corresponds to the switching sequence $2, 1, 2, 1, \ldots$. Notice that when $\rho > 0.5$, there are also other destabilizing sequences where the unstable mode (with subsystem matrix A_2) runs most of the time.

On the other hand, for $\rho < 0.5$, J_2 can be obtained negative, since in all optimal solutions, we have $\rho_{(1,1)} > 0$. Notice that the sequence $(1, 1)$ indicates the pattern where the stable mode (with subsystem matrix A_1) is active at consecutive time instants. For example, when $\rho = 0.48$, with $\varepsilon = 10^{-16}$, we obtain $J_2 = -0.7011$, indicating stability. Notice that when $\rho = 0.49$, with $\varepsilon = 10^{-16}$, we obtain a positive J_2 value, but by picking a smaller $\varepsilon = 10^{-24}$, we get $J_2 = -0.3166$ indicating stability.

VI. Conclusion

We explored almost sure asymptotic stability of a stochastic switched linear system. Our proposed stability analysis approach relies on studying the switched system's state at every $h \in \mathbb{N}$ steps. We obtained sufficient stability conditions and showed that the stability can be checked by solving a linear programming problem. The number of variables in this problem grows polynomially in the number of subsystems, but exponentially in h, which makes the computation difficult when h is large. To overcome this issue, we constructed an alternative linear programming problem, where the number of variables grows polynomially in both the number of subsystems and h. Even though the calculation of the coefficients in the alternative problem takes additional time, the solution is obtained faster compared to the original problem.

Our linear programming-based analysis approach allows us to check stability without relying on statistical information on the mode signal. In particular, the probability of mode switches and the stationary distributions associated with the modes are not needed for stability analysis. We applied our approach in exploring networked control systems under malicious jamming attacks. The technical challenge there is that the attackers' specific strategies are not available for analysis. By using our approach, we showed that stability can be guaranteed under all possible attack strategies when the long-run average number of network transmission failures satisfies certain conditions. In practice, our approach can be used by system operators to assess the safety of industrial processes. Specifically, our stability results can be utilized in identifying the level of jamming attacks that can be tolerated on the communication channels used for the measurement and the control of a plant.

Investigations of the case with noisy dynamics and the stabilization problem are part of our future extensions. In the stabilization problem, the controller may not have access to precise information of the active mode. In those cases, the system modes are divided into several groups, and the controller only knows which group contains the currently active mode. This problem was considered for discrete- and continuous-time Markov jump systems by [56], [57]. For this problem setting, our approaches may be extended for the case where the mode signal is not necessarily a Markov process.

References

[1] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Syst.*, vol. 19, no. 5, pp. 59–70, 1999.

[2] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: A survey of recent results," *IEEE Trans. Autom. Control*, vol. 54, no. 2, pp. 308–322, 2009.

[3] D. Liberzon, J. P. Hespanha, and A. S. Morse, "Stability of switched systems: A Lie-algebraic condition," *Syst. Control Lett.*, vol. 37, no. 3, pp. 117–122, 1999.

[4] J. Daafouz, P. Riedinger, and C. Iung, "Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach," *IEEE Trans. Autom. Control*, vol. 47, no. 11, pp. 1883–1887, 2002.

[5] D. Liberzon, *Switching in Systems and Control*. Birkhauser, 2003.

[6] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proc. IEEE Conf. Dec. Contr.*, pp. 2655–2660, 1999.

[7] G. Zhai, H. Guisheng, B. Hu, K. Yasuda, and A. N. Michel, "Stability analysis of switched systems with stable and unstable subsystems: An average dwell time approach," *Int. J. Syst. Sci.*, vol. 32, no. 8, pp. 1055–1061, 2001.

[8] H. Zhang, D. Xie, H. Zhang, and G. Wang, "Stability analysis for discrete-time switched systems with unstable subsystems by a mode-dependent average dwell time approach," *ISA Trans.*, vol. 53, no. 4, pp. 1081–1086, 2014.

[9] H. Ishii, T. Basar, and R. Tempo, "Randomized algorithms for synthesis of switching rules for multimodal systems," *IEEE Trans. Autom. Control*, vol. 50, no. 6, pp. 754–767, 2005.

[10] A. R. Teel, A. Subbaraman, and A. Sferlazza, "Stability analysis for stochastic hybrid systems: A survey," *Automatica*, vol. 50, no. 10, pp. 2435–2456, 2014.

[11] P. Shi and F. Li, "A survey on Markovian jump systems: Modeling and design," *Int. J. Control Autom. Syst.*, vol. 13, no. 1, pp. 1–16, 2015.

[12] Y. Fang, K. Loparo, and X. Feng, "Stability of discrete time jump linear systems," *J. Math. Systems Estim. Control*, vol. 5, no. 3, pp. 275–321, 1995.

[13] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear Systems*. Springer, 2004.

[14] W. Zhou, Q. Zhu, P. Shi, H. Su, J. Fang, and L. Zhou, "Adaptive synchronization for neutral-type neural networks with stochastic perturbation and Markovian switching parameters," *IEEE Trans. Cybern.*, vol. 44, no. 12, pp. 2848–2860, 2014.

[15] Q. Zhu, "Razumikhin-type theorem for stochastic functional differential equations with Lévy noise and Markov switching," *Int. J. Control*, vol. 90, no. 8, pp. 1703–1712, 2017.

[16] B. Wang and Q. Zhu, "Stability analysis of Markov switched stochastic differential equations with both stable and unstable subsystems," *Syst. Control Lettr.*, vol. 105, pp. 55–61, 2017.

[17] P. Bolzern, P. Colaneri, and G. De Nicolao, "Markov jump linear systems with switching transition rates: Mean square stability with dwell-time," *Automatica*, vol. 46, no. 6, pp. 1081–1088, 2010.

[18] P. Bolzern, P. Colaneri, and G. De Nicolao, "Design of stabilizing strategies for discrete-time dual switching linear systems," *Automatica*, vol. 69, pp. 93–100, 2016.

[19] H. Shisheh Foroush and S. Martínez, "On single-input controllable linear systems under periodic DoS jamming attacks," in *Proc. SIAM Conf. Contr. Appl.*, 2013.

[20] C. De Persis and P. Tesi, "Input-to-state stabilizing control under denial-of-service," *IEEE Trans. Autom. Control*, vol. 60, no. 11, pp. 2930–2944, 2015.

[21] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Networked control under random and malicious packet losses," *IEEE Trans. Autom. Control*, vol. 62, no. 5, pp. 2434–2449, 2017.

[22] W. Xu, W. Trappe, Y. Zhang, and T. Wood, "The feasibility of launching and detecting jamming attacks in wireless networks," in *Proc. 6th ACM Int. Symp. Mobile Ad Hoc Network. Comput.*, pp. 46–57, 2005.

[23] S. Amin, A. A. Cárdenas, and S. S. Sastry, "Safe and secure networked control systems under Denial-of-Service attacks," in *Proc. 12th HSCC*, pp. 31–45, 2009.

[24] K. Pelechrinis, M. Iliofotou, and S. V. Krishnamurty, "Denial of service attacks in wireless networks: The case of jammers," *IEEE Commun. Surveys Tuts.*, vol. 13, no. 2, pp. 245–257, 2011.

[25] A. T. Mizrak, S. Savage, and K. Marzullo, "Detecting malicious packet losses," *IEEE Trans. Parallel Distrib. Syst.*, vol. 20, no. 2, pp. 191–206, 2009.

[26] T. Shu and M. Krunz, "Privacy-preserving and truthful detection of packet dropping attacks in wireless ad hoc networks," *IEEE Trans. Mobile Computing*, vol. 14, no. 4, pp. 813–828, 2015.

[27] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Event-triggered control over unreliable networks subject to jamming attacks," in *Proc. IEEE Conf. Dec. Contr.*, pp. 4818–4823, 2015.

[28] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Event-triggered output feedback control resilient against jamming attacks and random packet losses," in *Proc. IFAC NecSys*, pp. 270–275, 2015.

[29] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Random and malicious packet transmission failures on multi-hop channels in networked control systems," in *Proc. IFAC NecSys*, pp. 49–54, 2016.

[30] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Wireless control under jamming attacks with bounded average interference power," in *Proc. IFAC World Cong.*, pp. 8735–8740, 2017.

[31] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proc. IEEE*, vol. 95, no. 1, pp. 138–172, 2007.

[32] H. Ishii, "Limitations in remote stabilization over unreliable channels without acknowledgements," *Automatica*, vol. 45, no. 10, pp. 2278–2285, 2009.

[33] L. Xie and L. Xie, "Stability analysis of networked sampled-data linear systems with Markovian packet losses," *IEEE Trans. Autom. Control*, vol. 54, no. 6, pp. 1375–1381, 2009.

[34] K. Okano and H. Ishii, "Stabilization of uncertain systems with finite data rates and Markovian packet losses," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 4, pp. 298–307, 2014.

[35] A. Polanski, "Lyapunov function construction by linear programming," *IEEE Trans. Autom. Control*, vol. 42, no. 7, pp. 1013–1016, 1997.

[36] X. Liu, "Stability analysis of switched positive systems: A switched linear copositive Lyapunov function method," *IEEE Trans. Circuits Syst. II, Exp. Briefs.*, vol. 56, no. 5, pp. 414–418, 2009.

[37] R. Baier, L. Grüne, and S. Hafstein, "Linear programming based Lyapunov function computation for differential inclusions," *Discrete Contin. Dyn. Syst. Ser. B*, vol. 17, no. 1, pp. 33–56, 2012.

[38] S. Zhu, Q.-L. Han, and C. Zhang, "l1-gain performance analysis and positive filter design for positive discrete-time Markov jump linear systems: A linear programming approach," *Automatica*, vol. 50, no. 8, pp. 2098–2107, 2014.

[39] P. Bolzern, P. Colaneri, and G. De Nicolao, "On almost sure stability of discrete-time Markov jump linear systems," in *Proc. IEEE Conf. Dec. Contr*, pp. 3204–3208, 2004.

[40] Y. Huang, J. Luo, T. Huang, and M. Xiao, "The set of stable switching sequences for discrete-time linear switched systems," *J. Math. Anal. Appl.*, vol. 377, no. 2, pp. 732–743, 2011.

[41] J.-W. Lee and G. E. Dullerud, "Uniform stabilization of discrete-time switched and Markovian jump linear systems," *Automatica*, vol. 42, no. 2, pp. 205–218, 2006.

[42] M. Philippe, R. Essick, G. E. Dullerud, and R. M. Jungers, "Stability of discrete-time switching systems with constrained switching sequences," *Automatica*, vol. 72, pp. 242–250, 2016.

[43] B. C. Levy and M. Zorzi, "A contraction analysis of the convergence of risk-sensitive filters," *SIAM J. Control Optim.*, vol. 54, no. 4, pp. 2154–2173, 2016.

[44] M. Zorzi and B. C. Levy, "On the convergence of a risk sensitive like filter," in *Proc. IEEE Conf. Dec. Contr.*, pp. 4990–4995, 2015.

[45] P. Sadeghi, R. A. Kennedy, P. B. Rapajic, and R. Shams, "Finite-state Markov modeling of fading channels," *IEEE Signal Process. Mag.*, vol. 25, no. 5, pp. 57–80, 2008.

[46] M. Ellis, D. P. Pezaros, T. Kypraios, and C. Perkins, "A two-level Markov model for packet loss in UDP/IP-based real-time video applications targeting residential users," *Comput. Netw.*, vol. 70, no. 9, pp. 384–399, 2014.

[47] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Enhanced stability analysis for networked control systems under random and malicious packet losses," in *Proc. IEEE Conf. Dec. Contr.*, pp. 2721–2726, 2016.

[48] A. Hassibi, S. P. Boyd, and J. P. How, "Control of asynchronous dynamical systems with rate constraints on events," in *Proc. IEEE Conf. Dec. Contr.*, pp. 1345–1351, 1999.

[49] W. Zhang, M. S. Branicky, and S. M. Phillips, "Stability of networked control systems," *IEEE Contr. Syst. Mag.*, vol. 21, no. 1, pp. 84–99, 2001.

[50] J. Norris, *Markov Chains*. Cambridge University Press, 2009.

[51] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.

[52] A. Karr, *Probability*. Springer, 1993.

[53] J. S. Rosenthal, *A First Look at Rigorous Probability Theory*. World Scientific Publishing, 2011.

[54] M. Vidyasagar, *Hidden Markov Processes: Theory and Applications to Biology*. Princeton University Press, 2014.

[55] B. Korte and J. Vygen, *Combinatorial Optimization*. Springer, 2012.

[56] J. B. Do Val, J. C. Geromel, and A. P. C. Gonçalves, "The H2-control for jump linear systems: cluster observations of the Markov state," *Automatica*, vol. 38, no. 2, pp. 343–349, 2002.

[57] D. Li, D. Zhang, and H. Ji, "Stabilization of jump linear systems with partial observation of Markov mode," *Int. J. Pure Appl. Math.*, vol. 27, pp. 31–38, 2006.

Appendix

The following result provides lower- and upper-bounds for the long-run average of the product of two binary-valued processes.

Lemma A.1: For all binary-valued processes $\{\xi_1(t) \in \{0,1\}\}_{t\in\mathbb{N}_0}$ and $\{\xi_2(t) \in \{0,1\}\}_{t\in\mathbb{N}_0}$ that satisfy

$$\liminf_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \xi_i(t) \geq \varsigma_i, \tag{74}$$

$$\limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \xi_i(t) \leq \varrho_i, \quad i \in \{1,2\}, \tag{75}$$

almost surely with $\varsigma_i, \varrho_i \in [0,1]$, $i \in \{1,2\}$, we have

$$\liminf_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \xi_1(t)\xi_2(t) \geq \max\{0, \varsigma_1 + \varsigma_2 - 1\}, \tag{76}$$

$$\limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \xi_1(t)\xi_2(t) \leq \min\{\varrho_1, \varrho_2\}, \tag{77}$$

almost surely.

Proof: To show (76), first note that

$$\liminf_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \xi_1(t)\xi_2(t) = 1 - \limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \left(1 - \xi_1(t)\xi_2(t)\right). \tag{78}$$

For all $i, j \in \{1, 2\}$, $i \neq j$, we have

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} (1 - \xi_1(t)\xi_2(t)) \\
&\quad = \limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} (\xi_i(t)(1 - \xi_j(t)) + (1 - \xi_i(t))) \\
&\quad \leq \limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} (\xi_i(t)(1 - \xi_j(t))) \\
&\qquad + \limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} (1 - \xi_i(t)) . \qquad (79)
\end{aligned}$$

Since $\xi_i(t)(1-\xi_j(t)) \leq \xi_i(t)$ and $\xi_i(t)(1-\xi_j(t)) \leq 1-\xi_j(t)$, by (75), we have

$$\limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} (\xi_i(t)(1 - \xi_j(t))) \leq \limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \xi_i(t) \leq \varrho_i,$$

and by (74), we have

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} (\xi_i(t)(1 - \xi_j(t))) \leq \limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} (1 - \xi_j(t)) \\
&\quad = 1 - \liminf_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \xi_j(t) \leq 1 - \varsigma_j .
\end{aligned}$$

Therefore,

$$\limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} (\xi_i(t)(1 - \xi_j(t))) \leq \min\{\varrho_i, 1 - \varsigma_j\}. \qquad (80)$$

Furthermore, since $\limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} (1 - \xi_i(t)) \leq 1 - \varsigma_i$, it follows from (78)–(80) that

$$\begin{aligned}
&\liminf_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \xi_1(t)\xi_2(t) \geq 1 - (\min\{\varrho_i, 1 - \varsigma_j\} + 1 - \varsigma_i) \\
&\quad = \varsigma_i - \min\{\varrho_i, 1 - \varsigma_j\} = \max\{\varsigma_i - \varrho_i, \varsigma_i + \varsigma_j - 1\}. \qquad (81)
\end{aligned}$$

Now, noting that $\varsigma_i - \varrho_i \leq 0$, $i \in \{1, 2\}$, and $\liminf_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \xi_1(t)\xi_2(t) \geq 0$, almost surely, we have (76) from (81).

Next, we prove (77). Since $\xi_1(t)\xi_2(t) \leq \xi_1(t)$ and $\xi_1(t)\xi_2(t) \leq \xi_2(t)$, we obtain

$$\limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \xi_1(t)\xi_2(t) \leq \limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \xi_i(t), \qquad (82)$$

for $i \in \{1, 2\}$. It then follows from (75) that $\limsup_{k\to\infty} \frac{1}{k}\sum_{t=0}^{k-1} \xi_1(t)\xi_2(t) \leq \varrho_i$, $i \in \{1, 2\}$, almost surely, which then implies (77). □